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On Average-State Observability of Nonlinear Systems

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Optimal Sensor Selection (OSS) Problem

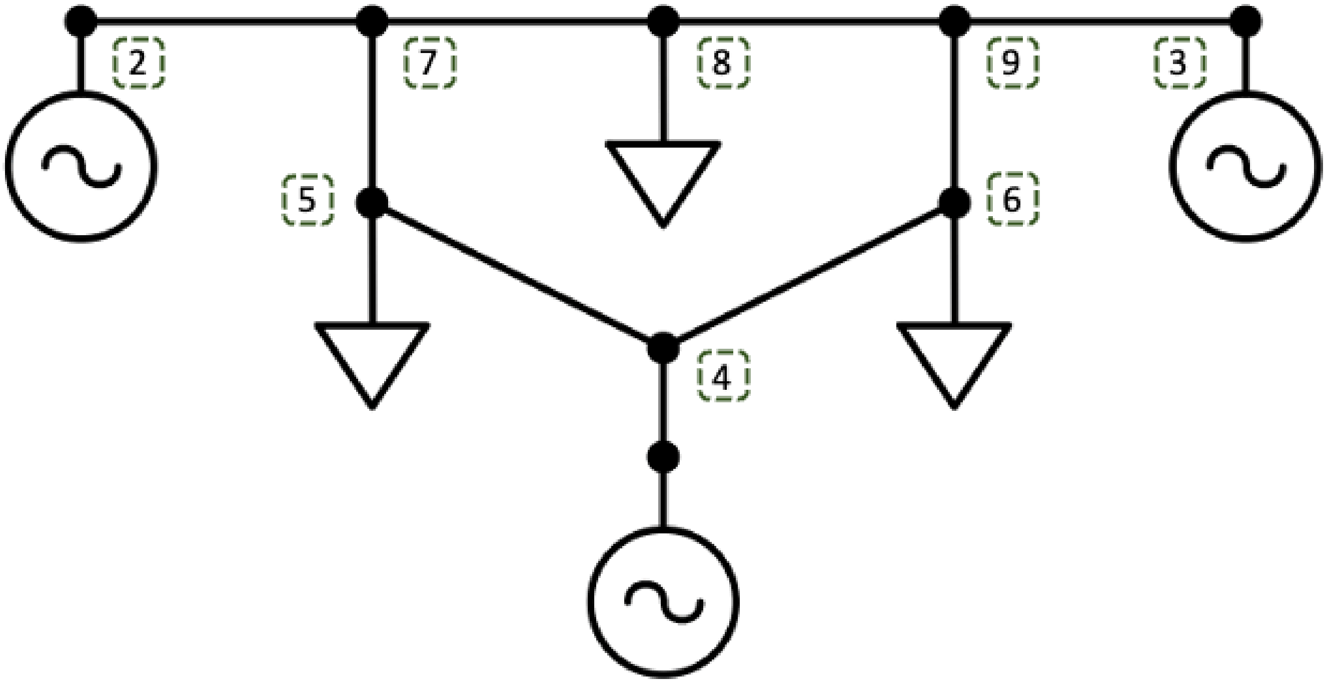
Obtain an *as observable system as possible* while using *a limited number of sensors* within the network

Power Networks

Water Networks

Transportation

Applications



Linear vs. Nonlinear System under OSS

Many methods for linear systems
Not as many for **nonlinear systems**

Linear Systems OSS

- Graph Theory — H. Zhang *et al.* 2017
- Semi-definite approximations — *Nugroho et al*. 2020
- Mixed Integer Programs — J. Taylor *et al.* 2017
- Heuristic Convex Approximation — Joshi & Boyd, 2009

Such methods perform poorly for *nonlinear systems*

Difficulty in *assessing the observability* of a nonlinear system

Observability of Dynamical Systems

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Evaluates the possibility of *inferring initial states* of a dynamical system by *monitoring its outputs*

Linear System

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$$

Observability Gramian

$$\boldsymbol{W}_{o,\text{linear}} = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

Extension to nonlinear systems is still an open problem

Observability of Nonlinear Systems

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State of the art and key weaknesses

- **System Linearization** → This measures observability is only valid around *operating regions* (A. Rouhani *et al.* 2017)
- **Lie Derivatives** → Results in *qualitative* observability measures that are complex to compute (S. Lall *et al.* *1999*; A. J. Krener and K. Ide, *2009*)
- **Empirical Gramian** → Not applicable to large-scale networks (A. Rouhani *et al.* 2017)

Commonly used approach

Observability Framework Utilized

Haber *et al.* 2018, IEEE TCNS

Observability based on concept of observability Matrix built under a moving horizon estimation (MHE) scheme

of work by Haber *et al.* 2018, to account for

As claimed by Haber *et al.*, this method is *most scalable for stiff non-linear dynamics*

Drawback: Local observability

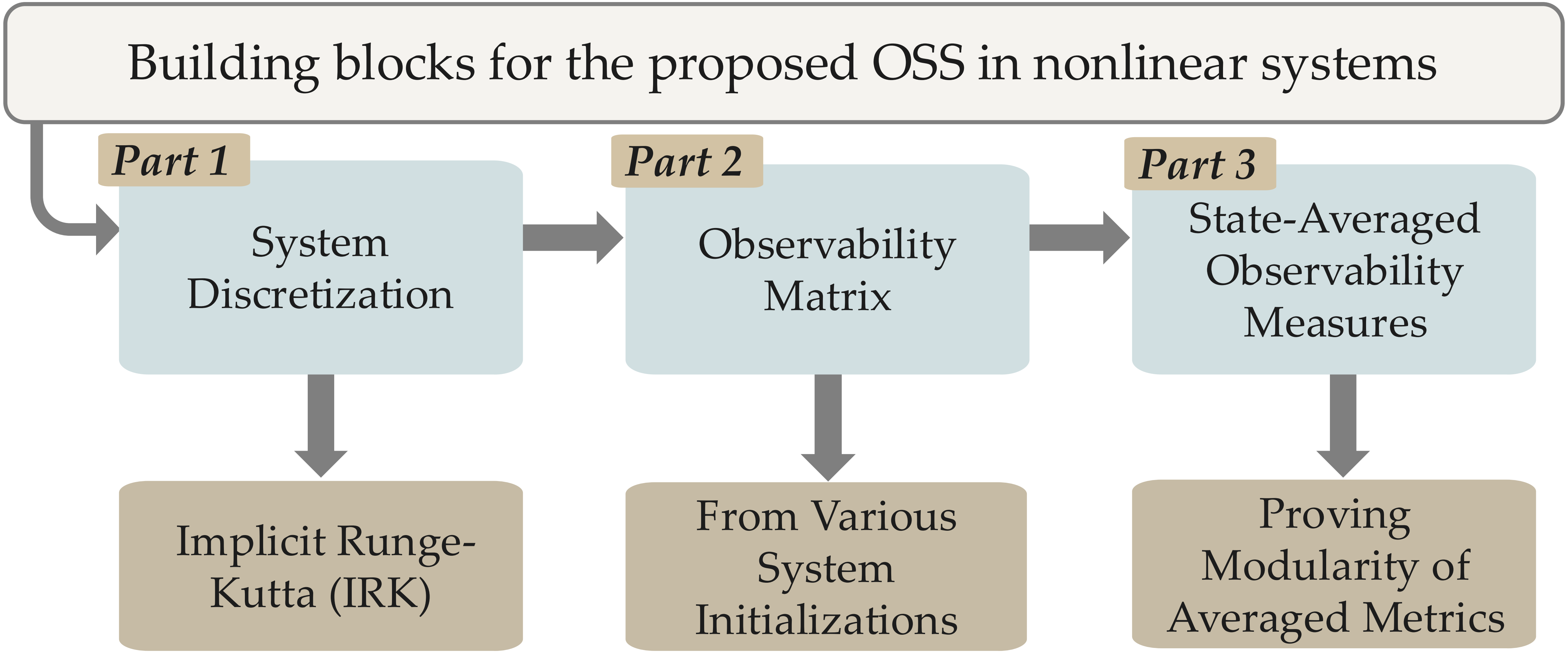
It depends on initial states that might be *unknown or uncertain; it results in different sensor selections*

Motivation: **State-averaged observability measures**

Paper Contributions

- *OSS that is robust towards* uncertain initial conditions
- *Scalable OSS* by proving retaining *submodularity of objective function*

Building Blocks of the Proposed OSS



Discretization: Implicit Runge-Kutta (IRK)

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CT domain

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$$
$$\boldsymbol{y}(t) = \boldsymbol{\Gamma}\boldsymbol{C}\boldsymbol{x}(t)$$

Sensor selection encoded in Γ

Discretization

The Gramian can be written as a function of $\boldsymbol{x}_0$

DT domain

- $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \frac{T}{4}\left(\boldsymbol{f}(\zeta_{1,k+1}) + 3\boldsymbol{f}(\zeta_{2,k+1})\right)$
- $\boldsymbol{x}_k := \boldsymbol{x}_k(\boldsymbol{x}_0)\ \forall\ k = \{0, 1, \ldots, \mathrm{N}-1\}$
- $\zeta_{1,k+1}, \zeta_{2,k+1} \in \mathbb{R}^{n_x}$ are auxiliary vectors for computing $\boldsymbol{x}_{k+1} \in \mathbb{R}^{n_x}$

A. Iserles, 2008

Nonlinear Observability Matrix

Observability matrix with sensor selection

$$\boldsymbol{W}(\gamma, \boldsymbol{x}_0) := \boldsymbol{J}_g^\top(\boldsymbol{x}_0)\boldsymbol{J}_g(\boldsymbol{x}_0)$$

Jacobian around initial state $\boldsymbol{x}_0$

$$\boldsymbol{J}_g(\gamma, \boldsymbol{x}_0) := \boldsymbol{J}_g(\boldsymbol{x}_0) = [\boldsymbol{I} \otimes \boldsymbol{\Gamma C}] \times \boldsymbol{\xi}(\boldsymbol{x}_0)$$

$$\boldsymbol{\xi}_i := \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{x}_0}$$

Based on one initial guess for $\boldsymbol{x}_0$

Partial derivatives along system trajectory

Based on observability matrix developed in Haber *et al.* 2018, IEEE TCNS

OSS Based on Maximizing System Observability

Objective: *Maximize system observability* while choosing a *fixed number of sensor nodes to placed or selected dynamically*

Scalar mapping function that have modular and submodular properties

(P1)

$$\begin{aligned} \underset{\gamma}{\text{maximize}} \quad & \mathcal{O}(\gamma) := \mathcal{L}\left(\boldsymbol{W}(\boldsymbol{x}_0)\right), \\ \text{subject to} \quad & \sum_{i=1}^{n_y} \gamma_i = r, \; \boldsymbol{\gamma} \in \{0,1\}^{n_y} \end{aligned}$$

$$\mathcal{L}(\cdot) = \text{trace}(\cdot)$$

$$\mathcal{L}(\cdot) = \text{log-det}(\cdot)$$

T. H. Summers *et al.* 2016, IEEE TCNS

Submodular Set Optimization for OSS

OSS *Mixed-Integer* Optimization Problem

(P1)
$$\underset{\boldsymbol{\gamma}}{\text{maximize}} \quad \mathcal{O}(\boldsymbol{\gamma})$$
$$\text{subject to} \quad \sum_{i=1}^{n_y} \gamma_i = r, \; \boldsymbol{\gamma} \in \{0,1\}^{n_y}$$

$\mathcal{S}$ encodes binary nodes selected γ

Greedy Approach

OSS as a *Submodular Set* Optimization Problem

(P2)
$$\mathcal{O}(\mathcal{S}) := \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{maximize}} \quad f(\mathcal{S})$$
$$\text{subject to} \quad |\mathcal{S}| = r$$

State-Averaged Observability Measures for OSS

Initial states are not known *a priori*, so the proposed observability matrix accounts for random initializations of q initial conditions

Parametrized observability matrix

$$\tilde{\boldsymbol{W}}^{(\kappa)}(\mathcal{S}, \hat{\boldsymbol{x}}_0^{(\kappa)}) := \sum_{j \in \mathcal{S}} \left(\sum_{i=0}^{\mathrm{N}-1} \left(\boldsymbol{\xi}_i^{(\kappa)}\right)^\top \boldsymbol{c}_j^\top \boldsymbol{c}_j \boldsymbol{\xi}_i^{(\kappa)} \right)$$

Derivatives along the trajectories from each system initialization

Submodular Functions

$$\frac{1}{q}\sum_{\kappa=1}^{q} \mathrm{trace}(\cdot)$$

$$\frac{1}{q}\sum_{\kappa=1}^{q} \mathrm{logdet}(\cdot)$$

State-averaged scaler mapping function

(P3)

$$\underset{\mathcal{S}}{\text{maximize}} \quad \mathcal{O}(\mathcal{S}) := \frac{1}{q}\sum_{\kappa=1}^{q} \mathcal{L}\left(\tilde{\boldsymbol{W}}^{(\kappa)}(\mathcal{S})\right)$$

$$\text{subject to} \quad |\mathcal{S}| = r, \; \mathcal{S} \subseteq \mathcal{V}$$

Plausibility under Submodularity Conditions

We show that *submodularity* of objective function is *retained based on Lemma 1.*

Modularity and submodularity of the original modular/submodular functions *is retained under a non-negative weighted sum*.

Lemma 1 *For set functions $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k : 2^{\mathcal{V}} \to \mathbb{R}$ that are submodular. Any conic combination, that is, any weighted non-negative sum defined as*

$$\mathcal{O}(\mathcal{S}) := \sum_{\kappa=1}^{q} w_k \mathcal{L}_k, \tag{1}$$

is submodular, such that $w_k \geq 0 \; \forall \; k$.

Consequence of Lemma 1

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We show that *submodularity* of objective function is *retained based on Lemma 1.*

Submodular Set Function is retained

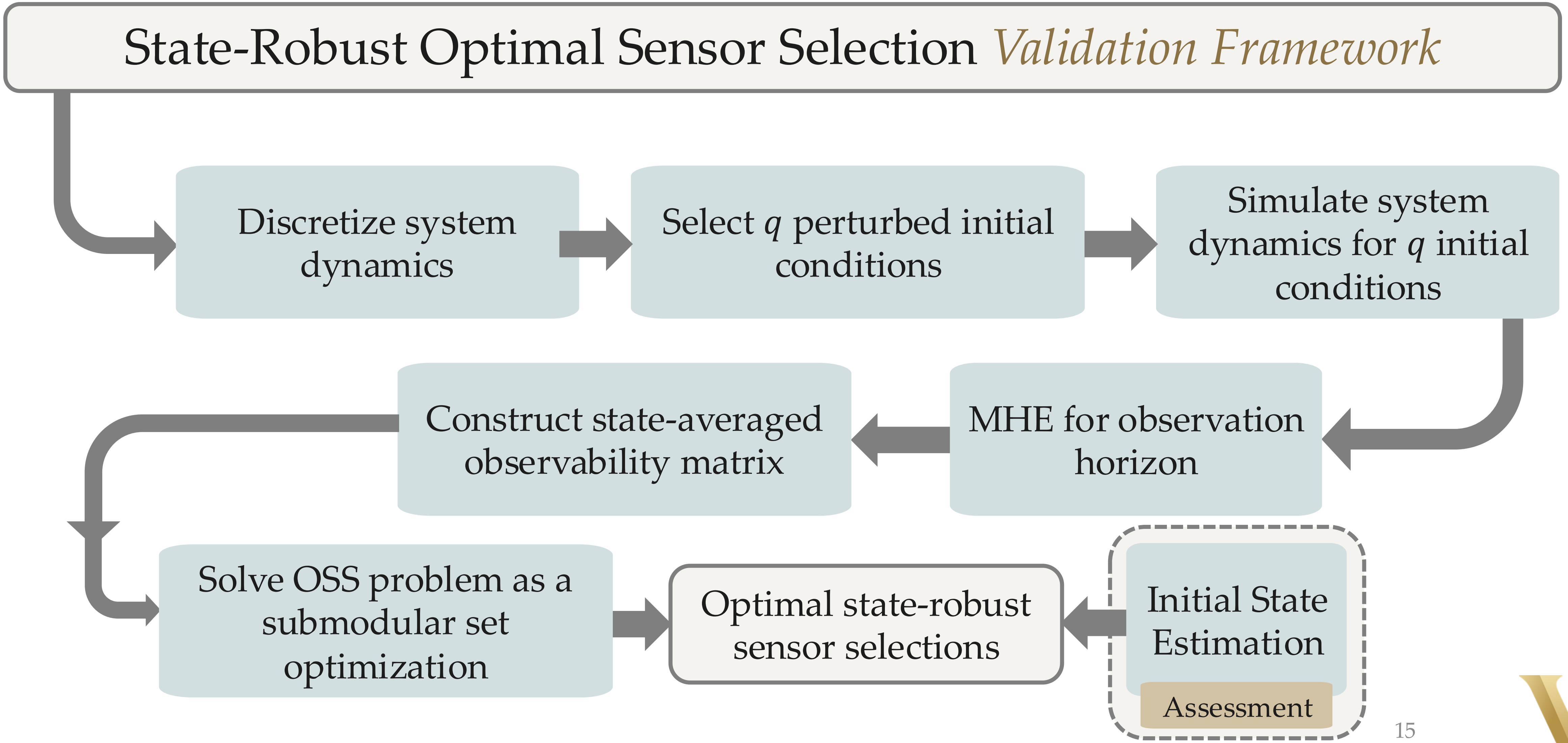
The OSS is rendered *scalable by using any greedy algorithm* to solve the OSS

Proposition 1 *Set functions* $\mathcal{O}_t : 2^{\mathcal{V}}$ *and* $\mathcal{O}_d : 2^{\mathcal{V}} \to \mathbb{R}$ *characterized by*

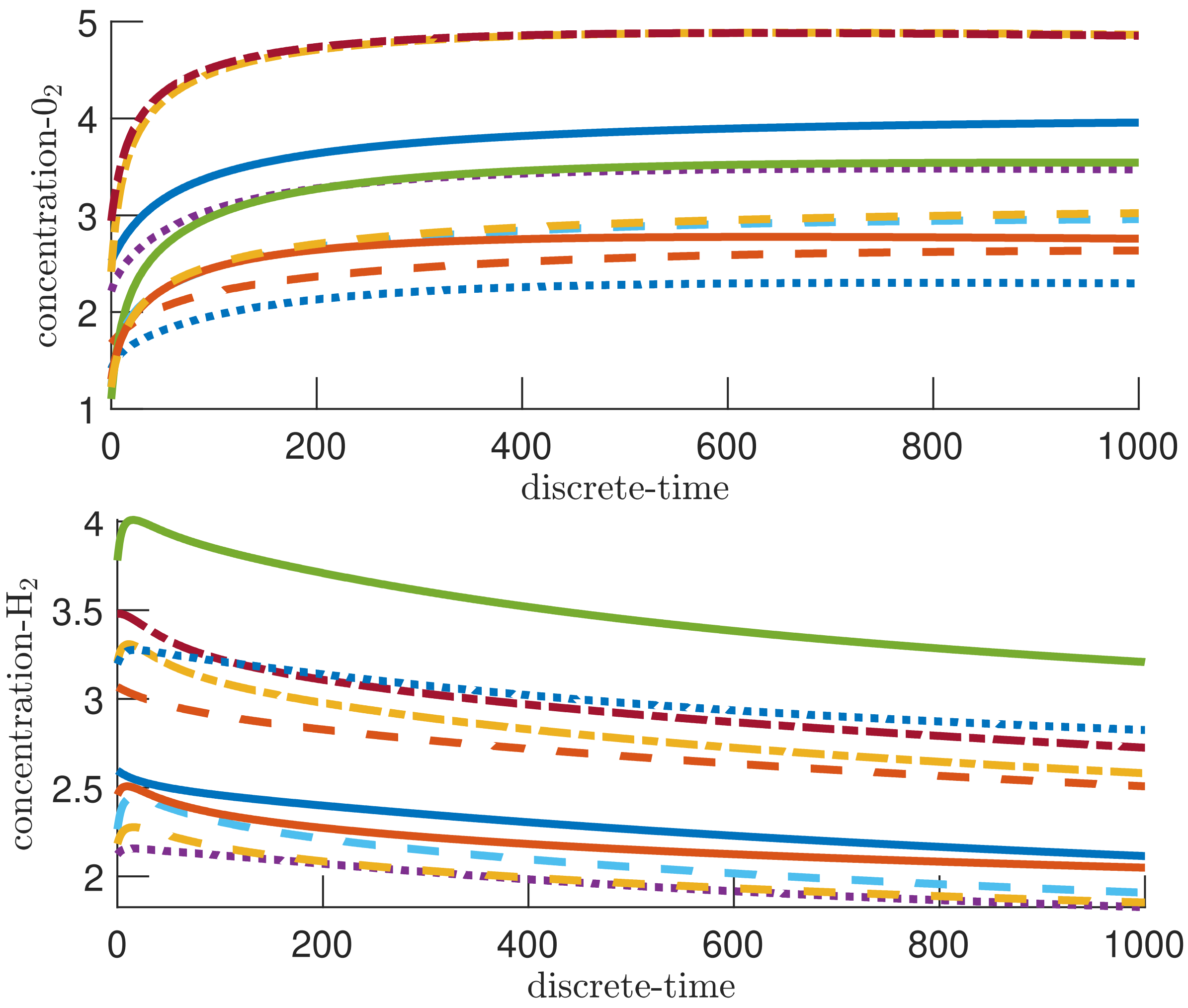
$$\mathcal{O}_t(\mathcal{S}) := \frac{1}{q}\sum_{\kappa=1}^{q} \text{trace}\left(\tilde{\boldsymbol{W}}^{(\kappa)}(\mathcal{S})\right), \quad \mathcal{O}_d(\mathcal{S}) := \frac{1}{q}\sum_{\kappa=1}^{q} \log\det\left(\tilde{\boldsymbol{W}}^{(\kappa)}(\mathcal{S})\right), \quad (1)$$

for $\mathcal{S} \subseteq \mathcal{V}$, *retain the modularity of original function for* $\mathcal{O}_t$ *and submodularity for* $\mathcal{O}_d$.

Roadmap Towards Numerical Validation



Combustion Reaction Networks: H_2/O_2



27 reactions and 9 species model

$$\dot{\boldsymbol{x}}(t) = \Theta\boldsymbol{\psi}\left(\boldsymbol{x}(t)\right)$$

Actual State

$$\boldsymbol{x} = [2,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0.2,\ 0]^T$$

Random System Initializations

Perturbing actual state

State trajectories for q = 10 initial conditions

Sensor Node Selection Diagram Comparison

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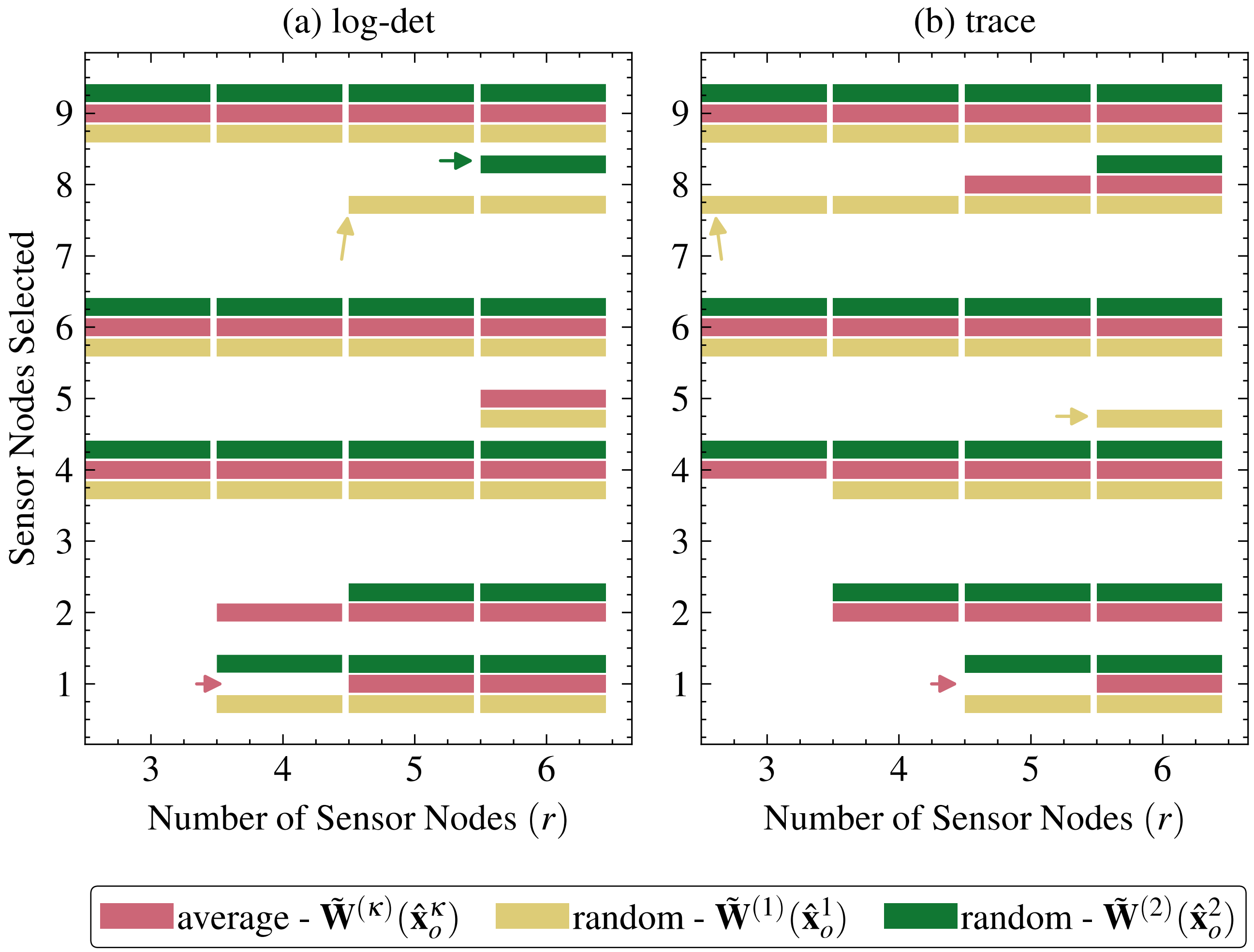


Nodes selected from proposed OSS

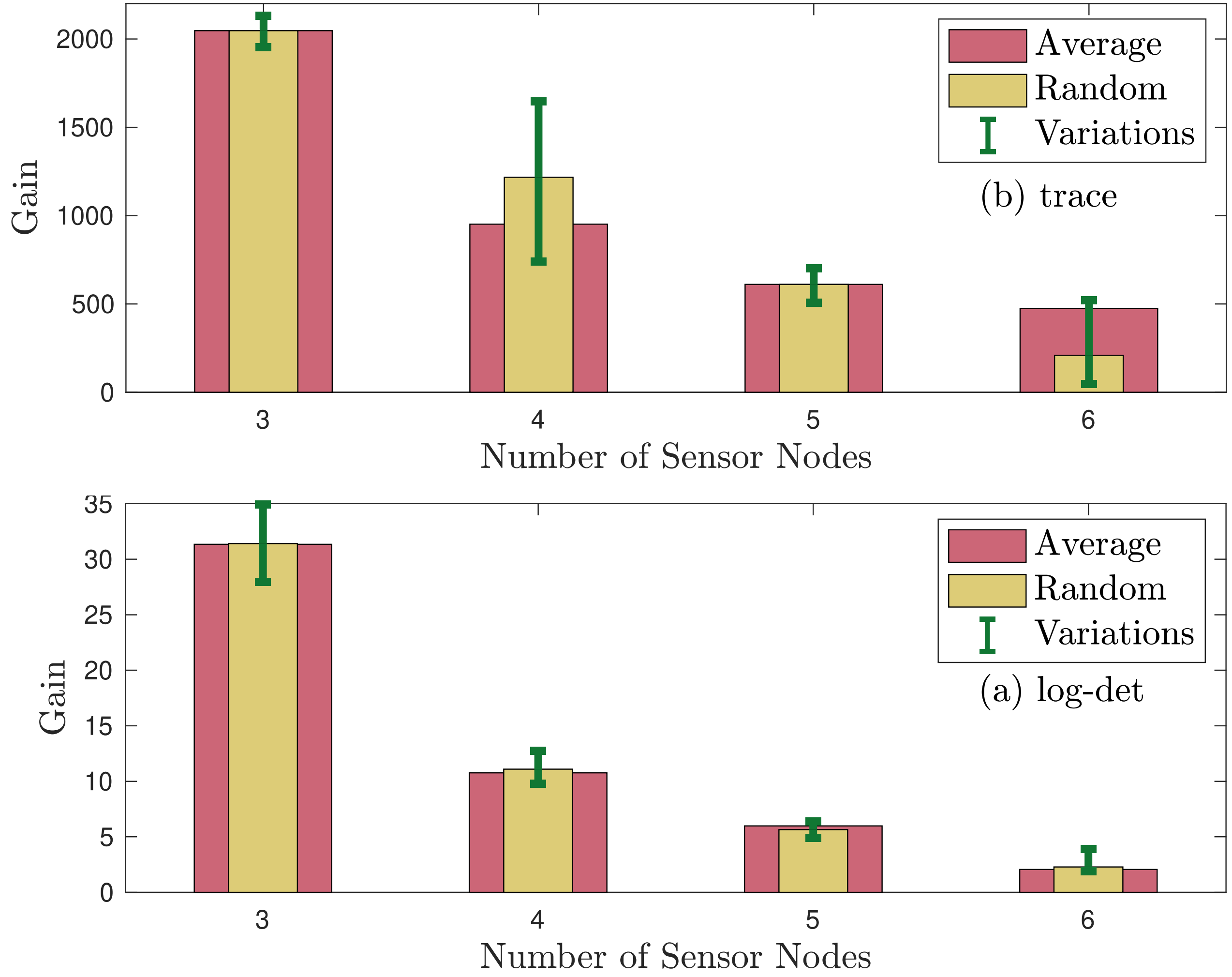
Solving the state-robust OSS vs. OSS based on single system initialization

Outcome

- Different *system trajectories and initial conditions* result in different OSS
- Chosen *sensor nodes* from the proposed OSS are *robust to different initial conditions*

Variability in Observability Gain

Observability measure gain based on single initialization vs. state-averaged



Outcomes

- *Attenuates variability* in observability measures
- *Observability Gain* within optimization instance becomes constant
- Sensor selection *robust* to initial conditions

Initial State Estimates: Based on OSS

(P4)

$$\underset{\hat{\boldsymbol{x}}_0 \in \boldsymbol{\mathcal{X}}}{\text{minimize}} \quad \boldsymbol{h}(\hat{\boldsymbol{x}}_0)^\top \boldsymbol{Q} \boldsymbol{h}(\hat{\boldsymbol{x}}_0)$$
$$\text{subject to} \quad \hat{\boldsymbol{x}}_0^l \leq \hat{\boldsymbol{x}}_0 \leq \hat{\boldsymbol{x}}_0^u$$

Lsqnonlin **solver**

Solved by implementing *trust-region-reflective algorithm*

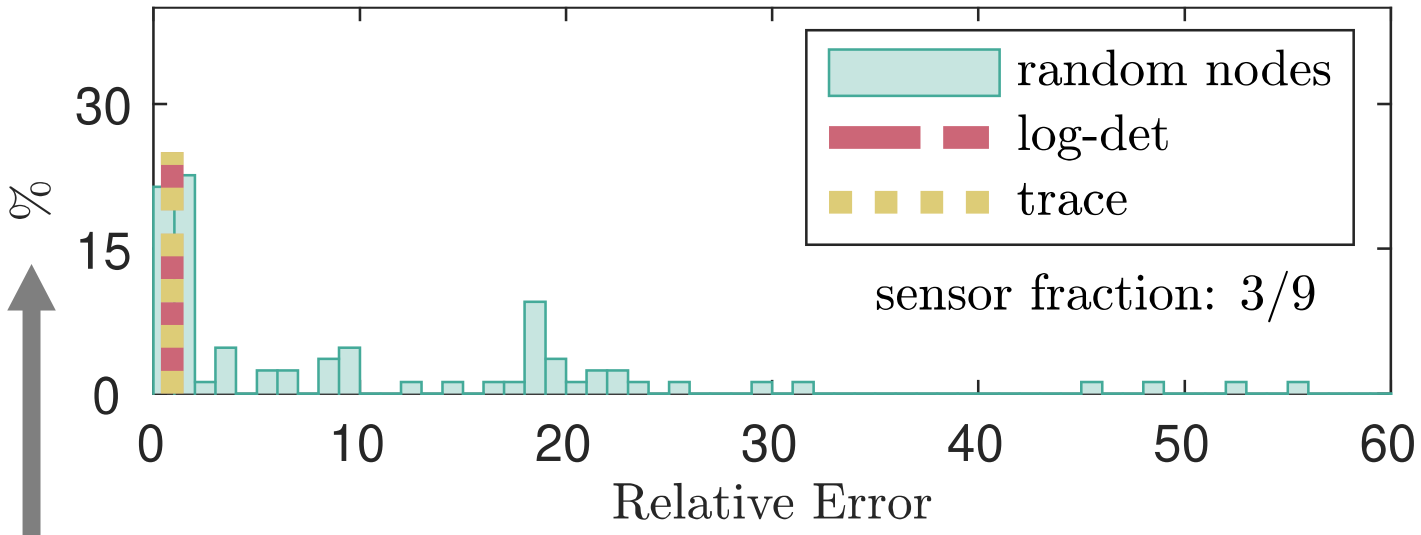
$\boldsymbol{Q} = \boldsymbol{I}$

Weight Matrix

$$\underbrace{\begin{bmatrix} \boldsymbol{h}_0(\hat{x}_0) \\ \boldsymbol{h}_1(\hat{x}_0) \\ \vdots \\ \boldsymbol{h}_{N-1}(\hat{x}_0) \end{bmatrix}}_{\boldsymbol{h}(\hat{x}_0)} := \underbrace{\begin{bmatrix} \tilde{\boldsymbol{y}}_0 \\ \tilde{\boldsymbol{y}}_1 \\ \vdots \\ \tilde{\boldsymbol{y}}_{N-1} \end{bmatrix}}_{\tilde{\boldsymbol{y}}} - \underbrace{\begin{bmatrix} \tilde{C}\hat{x}_1 \\ \tilde{C}\hat{x}_1 \\ \vdots \\ \tilde{C}\hat{x}_{N-1} \end{bmatrix}}_{\boldsymbol{g}(\hat{x}_0)}$$

Validation: Random Nodes Comparison

Percentage of random sensor locations with r=3



Relative Error: $e = \|\boldsymbol{x}_{\text{true}} - \hat{\boldsymbol{x}}\|_2 \,/\, \|\boldsymbol{x}_{\text{true}}\|_2$

On initial states estimates

we have 84 different random sensor mappings of size r=3

Outcomes

- Proposed Sensor Locations *achieve small initial state estimation errors*
- Random Node Selections *result in larger estimation errors*
- Validates the *optimality of the sensor locations* in terms of *system observability*

Future Work on Validating Proposed OSS

Additional questions regarding the robustness of state-averaged OSS framework

- Choice of *number of initial guesses* as a function of *number of internal states*
- Effect of the *variability of perturbed state trajectories* on observability
- Studying the proposed OSS on larger nonlinear systems
- Comparing OSS with the *Empirical Gramian*

Concluding Notes

A *state-averaged* optimal sensor selection framework for a nonlinear dynamical systems.

- *Extends* existing local observability OSS for nonlinear systems
- *Robust* against unknown/uncertain initial conditions
- *Retains* Submodularity → Computationally Efficient → *Scalable* OSS Formulation

Questions?

State-Robust Observability Measures for Sensor Selection in Nonlinear Dynamic Systems

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December 15th, 2023

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