VANDERBILT
School of Engineering

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Multilinear Extensions in Submodular Optimization for Optimal Sensor Scheduling in Nonlinear Networks

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Sensors Node Scheduling (SNS) in Networks

Sensing, Observability, and State Estimation as Foundations for Control

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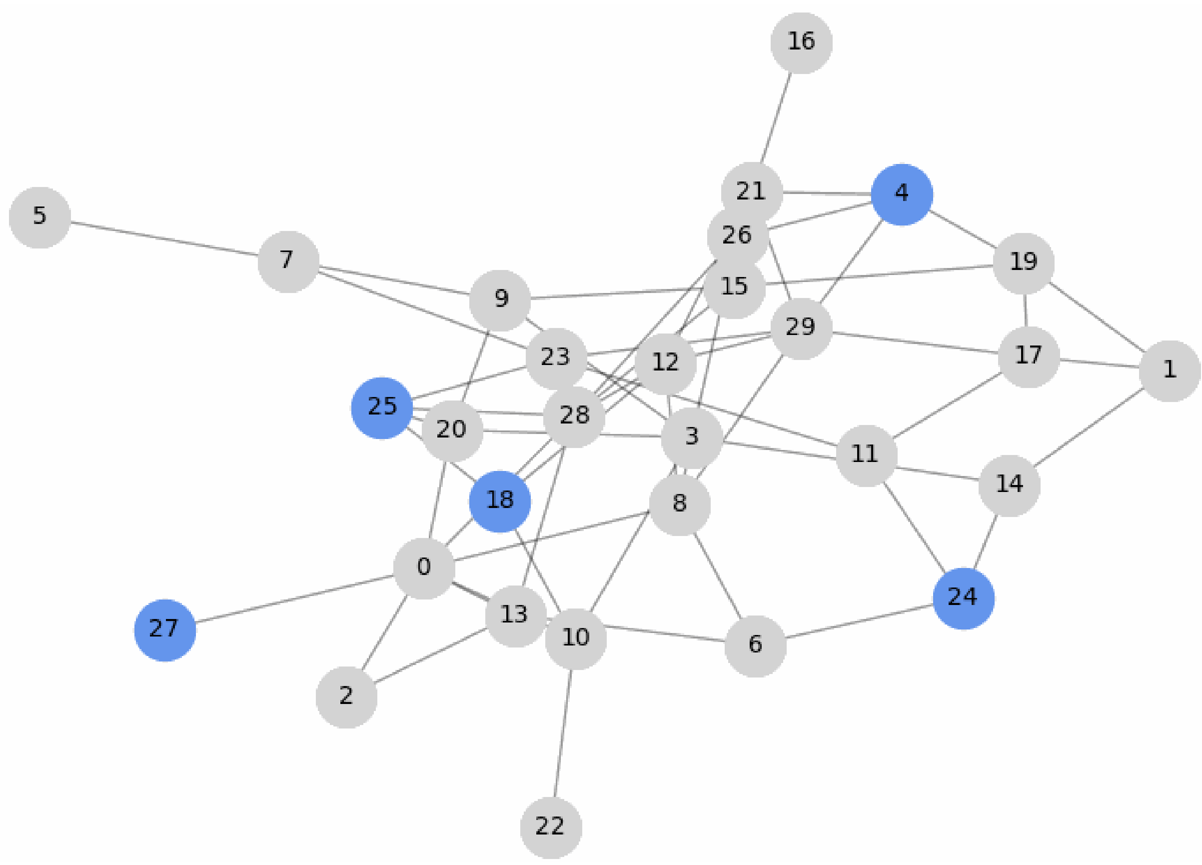
Essential for state-estimate
A first step is to enable accurate state-estimates

Combinatorial in Nature
Cannot handle larger systems

Suboptimal Solutions
Approximate problem formulations rendering it scalable



Applications

- Power Networks
- Transportation
- Water Networks
- Chemical Networks

The Sensor Scheduling Problem

A Submodular Optimization Approach for Sensor Allocation

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1 Submodular Maximization

$$f^* = \underset{\mathcal{S} \subseteq \mathcal{V},\ \mathcal{S} \in \mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

One definition of Submodularity

For any, $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V}$

$$f(\mathcal{A} \cup \{s\}) - f(\mathcal{A}) \geq f(\mathcal{B} \cup \{s\}) - f(\mathcal{B})$$

Kia, 2025

The Sensor Scheduling Problem

A Submodular Optimization Approach for Sensor Allocation

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$$f^* = \underset{\mathcal{S} \subseteq \mathcal{V},\, \mathcal{S} \in \mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

Submodular Maximization $\xrightarrow{\mathcal{I}}$

Matroid Constraint

$\mathcal{M} = (\mathcal{V}, \mathcal{I})$

$(i)\ \emptyset \in \mathcal{I}$

$(ii)\ A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$

$(iii)\ \mathcal{A}, \mathcal{B} \in \mathcal{I},\ |\mathcal{A}| < |\mathcal{B}|$
$\quad \exists s \in \mathcal{B} \setminus \mathcal{A},\ \mathcal{A} \cup \{s\} \in \mathcal{I}$

$\mathcal{I} = \mathcal{I}_c$ →

Uniform Matroid

$\mathcal{I}_c = \{S \subseteq \mathcal{V} : |S| = r\}$

(e.g., selecting r sensors from the entire network)

$\mathcal{I} = \mathcal{I}_p$ →

Partition Matroid

$\mathcal{V} = \bigcup_{i=1}^{\kappa} \mathcal{S}_i$

$\mathcal{I}_p = \{S \subseteq \mathcal{V} : |S \cap \mathcal{S}_i| \leq \kappa_i\}$

(e.g., selecting sensors with at most κ_i per subsystem)

Kia, 2025

Extensions in Submodular Optimization

Multilinear Extension for Submodular maximization

1 Submodular Maximization

2 Multilinear Extension

$$F_{\mathcal{S}}^* = \underset{\mathcal{S}\subseteq\mathcal{V},\ \mathcal{S}\in\mathcal{I}_c}{\text{maximize}} \quad F(\mathbf{x})$$

$F : [0,1]^n \rightarrow \mathbb{R}_{\geq 0}$

Inclusion probability

$$F(\mathbf{x}) = \sum_{\mathcal{S}\subset\mathcal{V}} f(\mathcal{S}) \prod_{s\in\mathcal{S}} [\mathbf{x}]_s \prod_{s\notin\mathcal{S}} (1 - [\mathbf{x}]_s)$$

How to compute?

Exclusion probability

Calinescu et al., 2009

Extensions in Submodular Optimization

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Multi-linear Extension for Submodular maximization

1 Submodular Maximization

2 Multilinear Extension

3 Computing the Extension

$$F(\mathbf{x}) = \sum_{\mathcal{S} \subset \mathcal{V}} f(\mathcal{S}) \prod_{s \in \mathcal{S}} [\mathbf{x}]_s \prod_{s \notin \mathcal{S}} (1 - [\mathbf{x}]_s)$$

Method of expectation

$$F(\mathbf{x}) = \mathbb{E}\left[f\left(\mathcal{S}_{\mathbf{x}}\right)\right]$$

Element s inclusion and exclusion in $\mathcal{S}$

Other methods

- Exact Methods: closed form expressions
- Approx. Method: Taylor series expansion

Iyer et al., 2014

Extensions in Submodular Optimization

Continuous Greedy Algorithm

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1 Submodular Maximization

2 Multilinear Extension

3 Computing the Extension

4 Algorithm

$$\underset{\mathcal{S}\in\mathcal{I}}{\operatorname{argmax}} \sum_{s\in\mathcal{S}} \mathbb{E}\left[f\left(\mathcal{S}_{\mathbf{x}} \cup \{s\}\right) - f\left(\mathcal{S}_{\mathbf{x}} \backslash \{s\}\right)\right]$$

- A path is defined: $\mathbf{x} : [0,1] \rightarrow \mathcal{S}_{\mathbf{x}}$
- The output is: $\mathbf{x}(1)$
- A discrete solution is then obtained using a pipage rounding algorithm

Chekuri and Quanrud, 2019

Observability-based SNS in Nonlinear Systems

Control-theoretic SNS problem

1 Observability-based SNS

Observability Gramian with sensor selection

$$\boldsymbol{W}(\gamma, \boldsymbol{x}_0)$$

$$f^* = \underset{\mathcal{S}\subseteq\mathcal{V},\, \mathcal{S}\in\mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

$$\text{trace}(\boldsymbol{W}(\cdot))$$
$$\log\det(\boldsymbol{W}(\cdot))$$
$$\text{rank}(\boldsymbol{W}(\cdot))$$

Modular and Submodular Properties

$\mathcal{S}$ encodes binary nodes selected γ

Summers et al., 2016

Observability-based SNS in Nonlinear Systems

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Gramian Computation for Nonlinear Systems

1 Observability-based SNS

2 Computing the Gramian

System Linearization	This measures observability is only valid around *operating regions* (A. Rouhani *et al.* 2017)
Lie Derivatives	*Qualitative* observability measures that are high order (S. Lall *et al. 1999*; A. J. Krener and K. Ide, *2009*)
Empirical Gramian	Not applicable to large-scale networks (A. Rouhani *et al.* 2017)

Observability-based SNS in Nonlinear Systems

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From Nonlinear to Variational Dynamics

1 Observability-based SNS

2 Computing the Gramian

3 Variational Dynamics

Kawano and Scherpen, 2019

Discrete Variational System Dynamics

$$\boldsymbol{x}_k = \boldsymbol{x}_{k-1} + \tilde{\boldsymbol{f}}\left(\boldsymbol{x}_{k-1}\right)$$

Variational Form

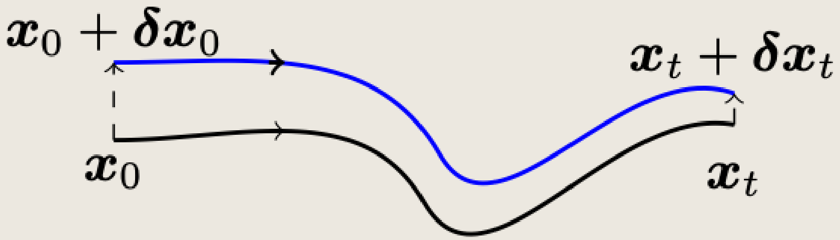
$$\boldsymbol{\delta x}_k = \boldsymbol{\Phi}_0^k\left(\boldsymbol{x}_0\right)\boldsymbol{\delta x}_0$$

Simply follows the system trajectory along the tangent space

Variational Mapping Function

$$\boldsymbol{\Phi}_0^k = \boldsymbol{\Phi}_{k-1}^k \boldsymbol{\Phi}_{k-2}^{k-1} \dots \boldsymbol{\Phi}_0^1 \boldsymbol{\Phi}_0^0 = \prod_1^{i=k} \boldsymbol{\Phi}_{i-1}^i$$

Computed along using chain rule



Observability-based SNS in Nonlinear Systems

Building the Variational Observability Gramian (Var-Gram)

1. Observability-based SNS
2. Computing the Gramian
3. Variational Dynamics
4. Variational Gramian

Kazma and Taha, 2024

Column vectors of $\boldsymbol{\Phi}_0^k$

Var-Gram $$\boldsymbol{V}_o(\mathcal{S}) = \sum_{j\in\mathcal{S}}\left(\sum_{i=0}^{N-1}\left(\boldsymbol{\varphi}_0^i\right)^\top \tilde{\boldsymbol{c}}_j^\top \tilde{\boldsymbol{c}}_j \boldsymbol{\varphi}_0^i\right)$$

Equivalent to LTI Gramian $$\boldsymbol{V}_o\left(\boldsymbol{x}_0\right) = \boldsymbol{W}\left(\boldsymbol{x}_0\right)$$

Equivalent to Empirical Gramian $$\boldsymbol{V}_o\left(\boldsymbol{x}_0\right) = \boldsymbol{W}_{\text{emp}}\left(\boldsymbol{x}_0\right)$$

Case Studies – Combustion Reaction Network

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System Setup and Parameters

$$\sum_{i=1}^{n_x} q_{ji}\mathcal{R}_i \rightleftarrows \sum_{i=1}^{n_x} w_{ji}\mathcal{R}_i$$

GRI30 - Methane Combustion

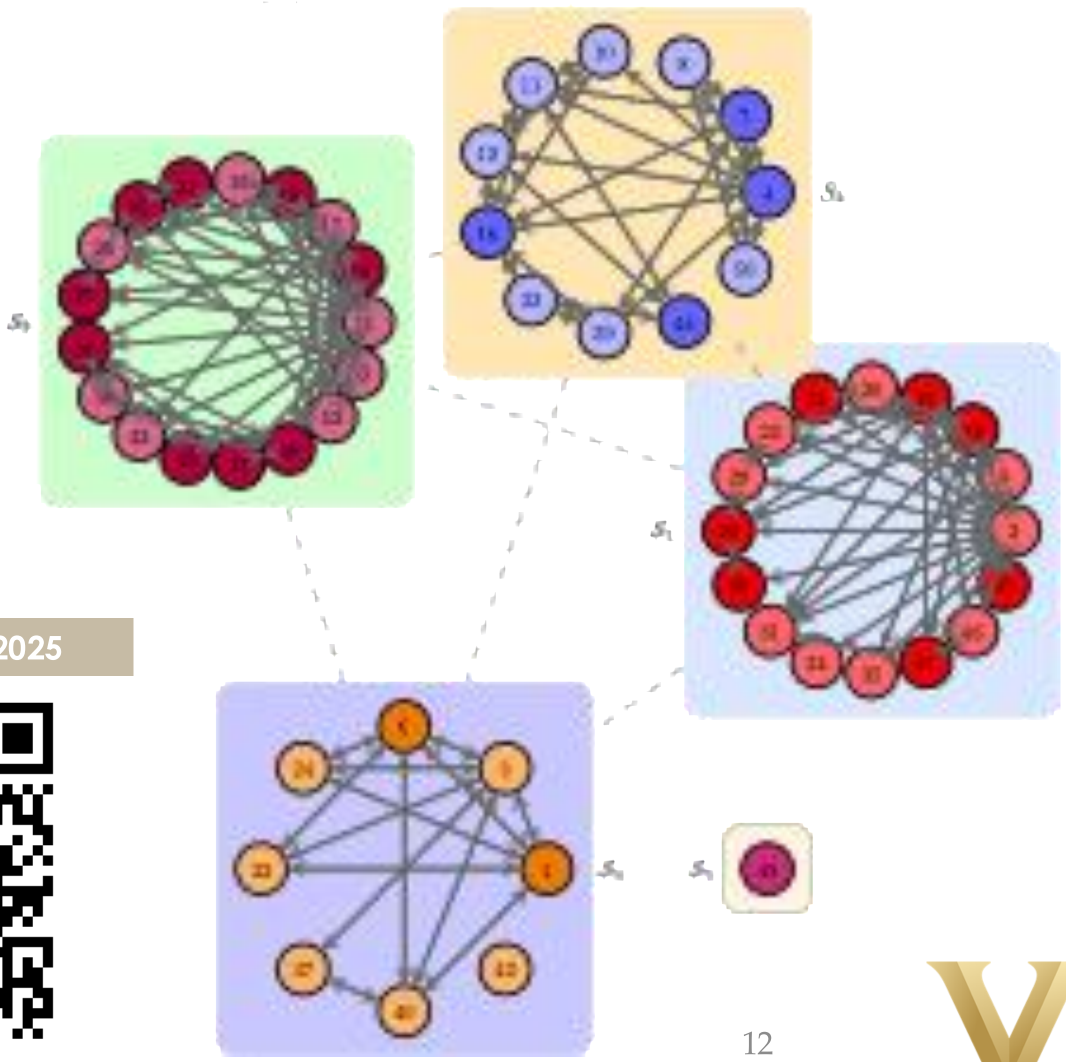
- $\dot{x}(t) = \Theta\psi\left(x(t)\right)$
- 325 chemical reactions
  53 species model

Parameters

- Observation Horizon
  N = 1000
- Initial Concentrations:

$$\boldsymbol{x}_0 = [0, 0, 0, 2, \cdots, 1, \cdots, 7.52, \cdots, 0]$$

Kazma and Taha, 2025



SNS State-estimation Error

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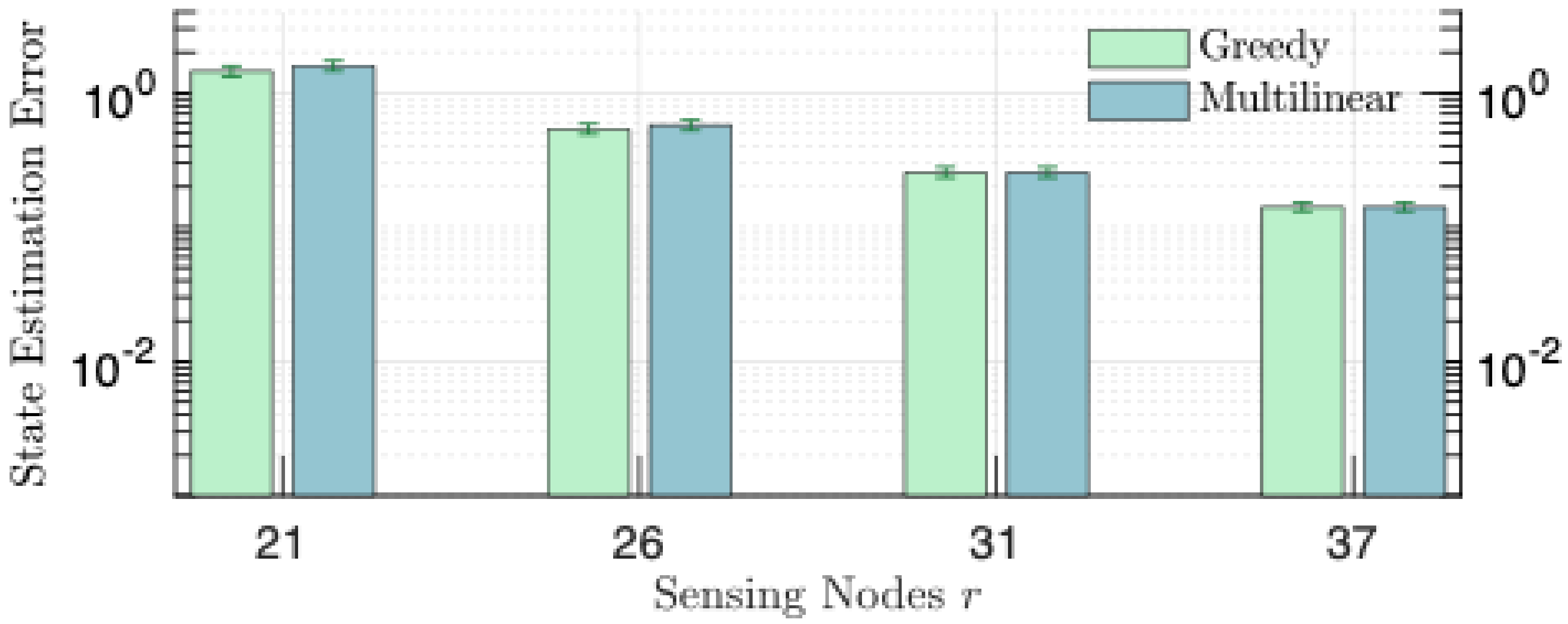
The optimality is the same
simple greedy Vs. the multilinear extension



Relative Error: $e = \|\boldsymbol{x}_{\text{true}} - \hat{\boldsymbol{x}}\|_2 \, / \, \|\boldsymbol{x}_{\text{true}}\|_2$

On initial states estimates

Concluding Notes and Future Directions

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Submodular Approach for SNS in Nonlinear Networks
Variational dynamics for Var-Gram computations

Multilinear Extensions for Observability-based Submodular SNS
Validated the use of such extension within the control-oriented framework

Application of multilinear extensions for distributed
Constraints that allocate sensors across partitioned systems

Application of New Parallel Algorithms
Faster methods that can handle even larger systems

Questions?

Multilinear Extensions in Submodular Optimization for Optimal Sensor Scheduling in Nonlinear Networks

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