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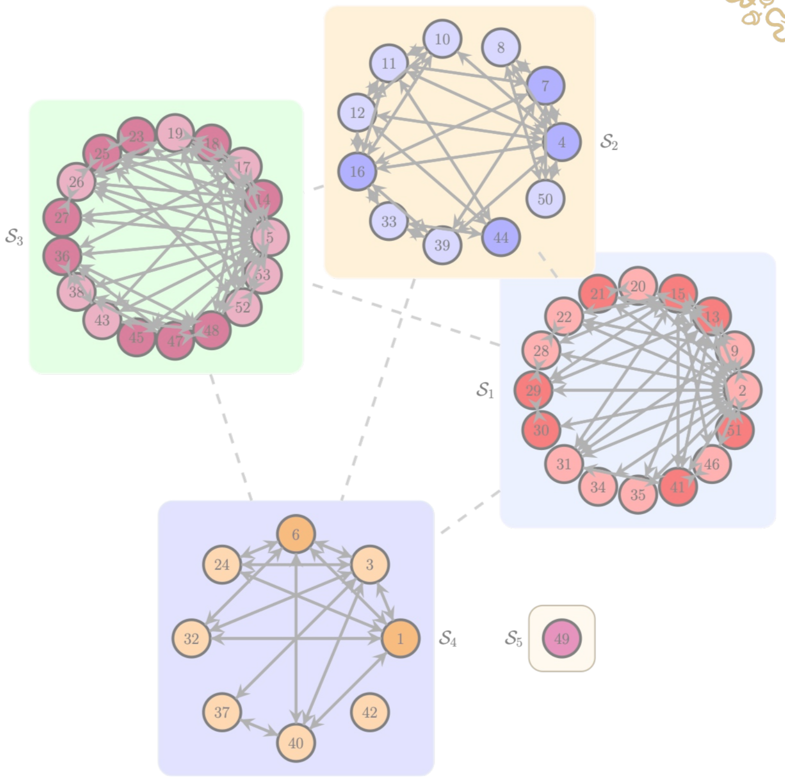
Scalable Control Engineering for Built and Natural Infrastructure Networks



Mohamad H. Kazma

June 8th, 2026

Critical Infrastructure Networks

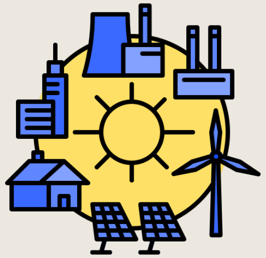
Resilience in the face of change and uncertainty

Power grid's stability
Intermittency in generation due to renewable energy resources (RERs)

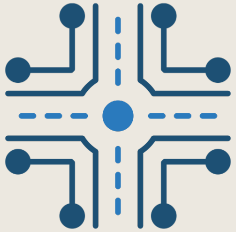
Water network water quality (WQ)
Increased water demand variability and uncertain WQ conditions

Overland Flow Monitoring
Climate induced variability leading to flash flooding

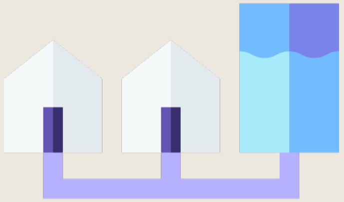
Natural and Built Infrastructures



Power Grid



Transportation Network



Drinking Water Networks

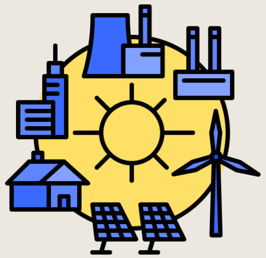
Watershed and River Networks

Critical Infrastructure Networks

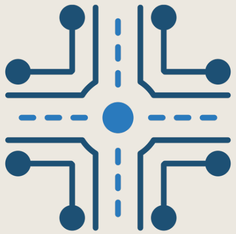
Technical challenges

- **High dimensionality, nonlinearity, and spatio-temporal behavior**
- **Uncertainty from demand variability and unpredictable human behavior**
- **Must adhere to physical conservation laws**

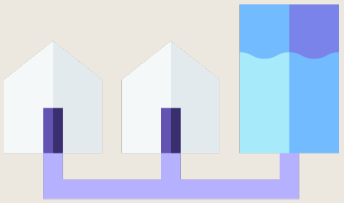
Natural and Built Infrastructures



Power Grid



Transportation Network



Drinking Water Networks

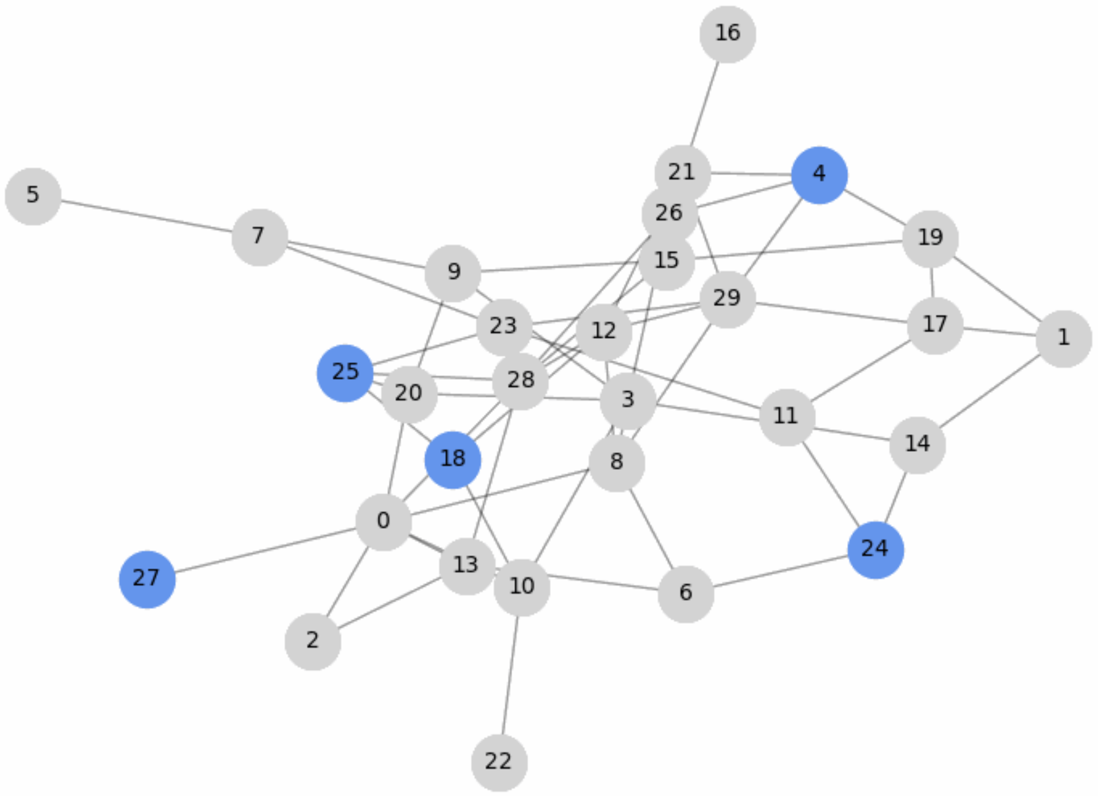
Watershed and River Networks

Monitoring of Infrastructure Networks

Because you cannot control what you do not observe: sensing comes first and the limitation of knowledge.

Why we need optimized sensing:

- For large-scale network, monitoring it all is nearly impossible
- Essential for state-estimate and feedback control



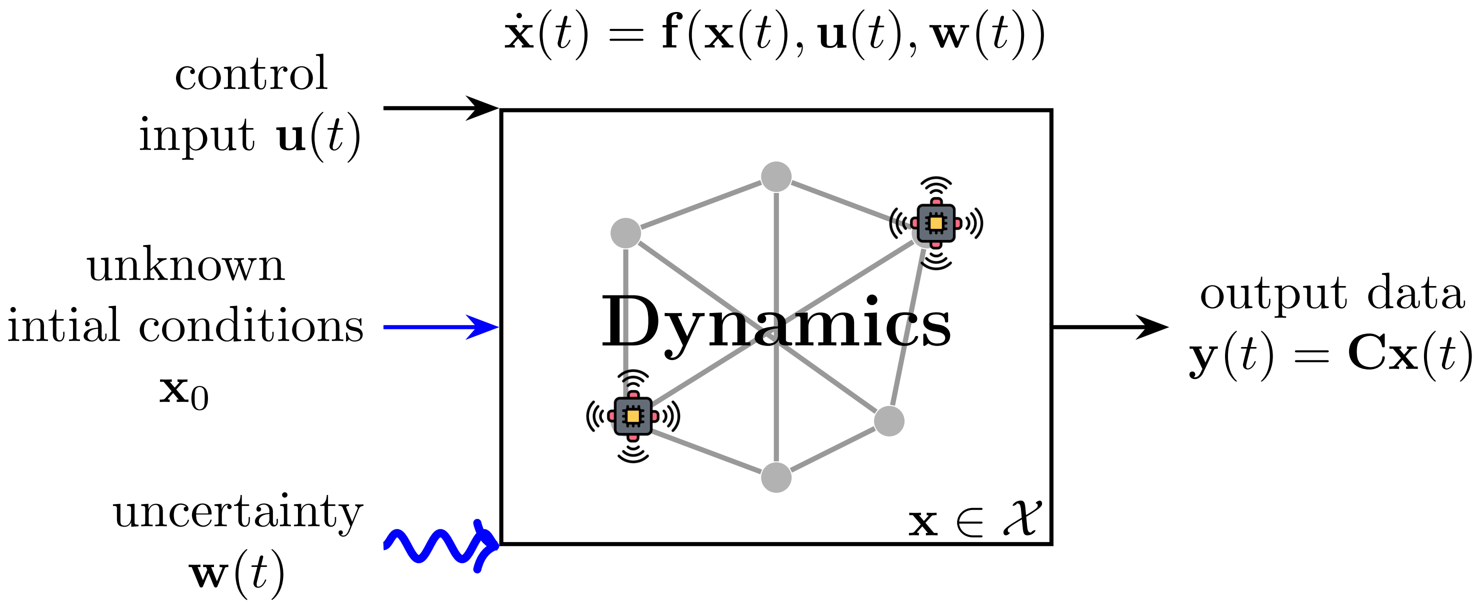
Technical challenges:

- Extracting information from measurements
- Optimal and Limited Sensor Allocation

Scalable theory for analyzing and informing decision-making in large-scale infrastructure networks

Monitoring of Dynamical Systems 101

From Lyapunov to Kalman and Modern System Theory



This is observability analysis; different communities call it different things

- **estimating** *x(t)* [large dimension]
- **from sensor data** *y(t)* [smaller dimension]
- under **uncertainty** *w(t)* and unknown initial conditions *x(0)*
- while **respecting** PDE/ODE **physics** *f(.)* and conservation laws

How is the literature Quantifying such Measures?

Linear system observability: good, scalable, useless for complex nonlinear

Nonlinear system observability: poor, uninterpretable, not scalable, used for toy examples

Linear Systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \in \mathbb{R}^{n_y}$$

- **Kalman's rank condition (Kalman 1960):**

$$\mathrm{rank}(\mathcal{O}) = n_x$$

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C}\boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n_x-1} \end{bmatrix}$$

Nonlinear Systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \in \mathbb{R}^{n_x}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) \in \mathbb{R}^{n_y}$$

- **Lie derivative (Local) rank condition (Hermann and Krener 1977):**

$$\mathrm{rank}\left(\mathcal{O}\right) = n_x$$

$$\mathcal{O} = \begin{bmatrix} \nabla \boldsymbol{h}(\boldsymbol{x}) \\ \nabla L_{\boldsymbol{f}} \boldsymbol{h}(\boldsymbol{x}) \\ \nabla L_{\boldsymbol{f}}^2 \boldsymbol{h}(\boldsymbol{x}) \\ \vdots \\ \nabla L_{\boldsymbol{f}}^{n_x-1} \boldsymbol{h}(\boldsymbol{x}) \end{bmatrix}$$

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- **Lie derivative (Local) rank condition (Hermann and Krener 1977):**

$$\text{rank}\left(\mathcal{O}\right) = n_x$$

$$L_{\mathbf{f}}^k \mathbf{h}(\mathbf{x}) = \underbrace{\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial}{\partial \mathbf{x}}\left(\cdots \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})\right)\mathbf{f}(\mathbf{x})\cdots\right)\mathbf{f}(\mathbf{x})\right)\mathbf{f}(\mathbf{x})}_{k \text{ times}},$$

Binary indicators / useless

How is the literature Quantifying such Measures?

Linear system observability: good, scalable, useless for complex nonlinear

Nonlinear system observability: poor, uninterpretable, not scalable, used for toy examples

Linear Systems

- **Linear Observability Gramian:**

$$\boldsymbol{W} = \int_0^T \mathcal{O}^\top \mathcal{O} dt \in \mathbf{R}^{n_x \times n_x}$$

- Still meaningless for nonlinear systems

Nonlinear Systems

- **Empirical Observability Gramian:**

$$\mathbf{W}^\varepsilon = \frac{1}{4\varepsilon^2}\int_0^T \left(\boldsymbol{\Delta}\mathbf{Y}^\varepsilon(t)\right)^\top \boldsymbol{\Delta}\mathbf{Y}^\varepsilon(t)\, dt \in \mathbf{R}^{n_x \times n_x}$$

$$\left[\mathbf{y}^{+1}(t) - \mathbf{y}^{-1}(t), \quad \cdots, \quad \mathbf{y}^{+n_x}(t) - \mathbf{y}^{-n_x}(t)\right]$$

- Computationally expensive
- Imagine perturbing and simulating a system $2n_x$ times

This work more precisely

Scalable theory with application for observability analysis and sensor allocation of nonlinear infrastructure networks

Today's talk

Part 1

1. New Observability Gramian
2. Some applications on the topic

Part 2

3. Partitioning Networks
4. Probabilistic Perspective

New Observability Gramian for Nonlinear Systems

1

Variational Gramian for local uniform observability

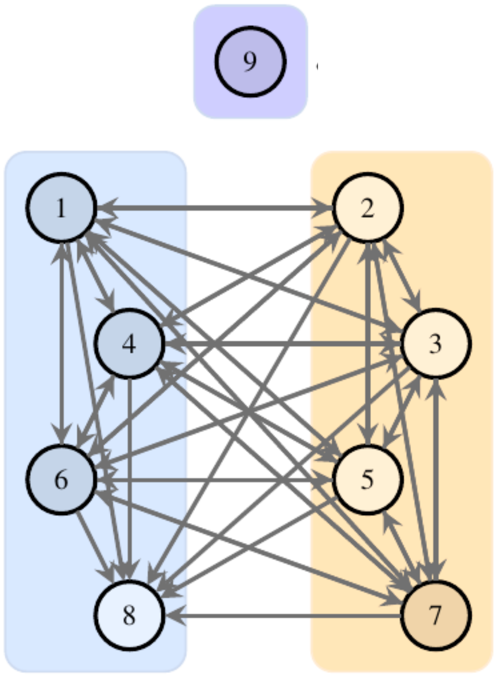
New Measures for Nonlinear Dynamic Systems

How can we measure system-theoretic properties

System-theoretic notions not easily transferable to complex, networked, and nonlinear systems

For Linear Systems
Measures do not work for nonlinear unless about a constant operating point

For Nonlinear Systems
Either Qualitative or Empirical methods that are terse to compute



$$\sum_{i=1}^{n_x} q_{ji}\mathcal{R}_i \rightleftarrows \sum_{i=1}^{n_x} w_{ji}\mathcal{R}_i$$

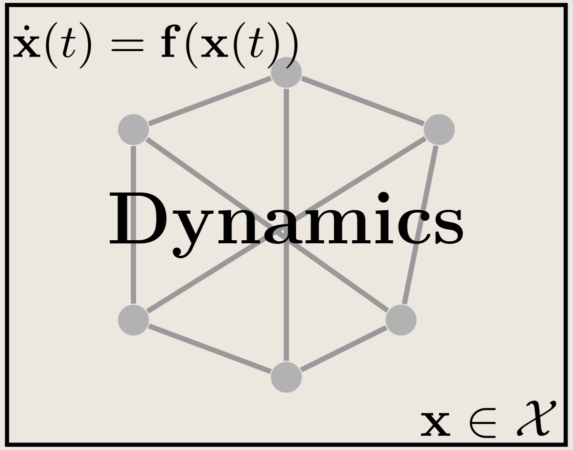
Variational Observability for Nonlinear Systems

Systems-theoretic observability quantification for nonlinear systems

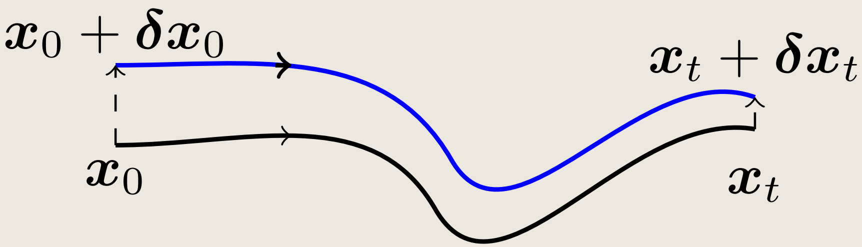
1 Variational Dynamics

Discrete-time Dynamics

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \tilde{\boldsymbol{f}}(\boldsymbol{x}_k)$$
$$\boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k)$$



Simply depicts how infinitesimal perturbations evolve along a trajectory



Variational Dynamics

$$\boldsymbol{\delta x}_{k+1} = \boldsymbol{\Phi}_0^k \boldsymbol{\delta x}_0$$
$$\boldsymbol{\delta y}_k = \boldsymbol{\Psi}_0^k \boldsymbol{\delta x}_0$$

Initial perturbation

Variational Mapping Function

$$\boldsymbol{\Phi}_0^k = \left(\boldsymbol{I}_{n_x} + \frac{\partial \tilde{\boldsymbol{f}}(\boldsymbol{x}_k)}{\partial \boldsymbol{x}_k}\right) \frac{\partial \boldsymbol{x}_k}{\partial \boldsymbol{x}_0}$$

References

A. Iserles, A First Course in the numerical analysis of differential equations, 2nd ed. Cambridge University Press, 2009.

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Systems-theoretic observability quantification for nonlinear systems

1 Variational Dynamics

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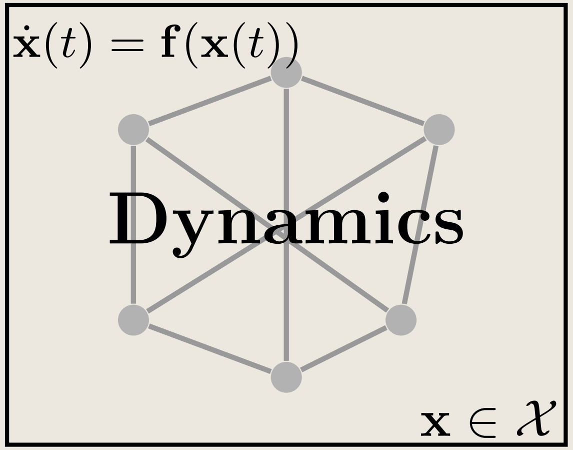

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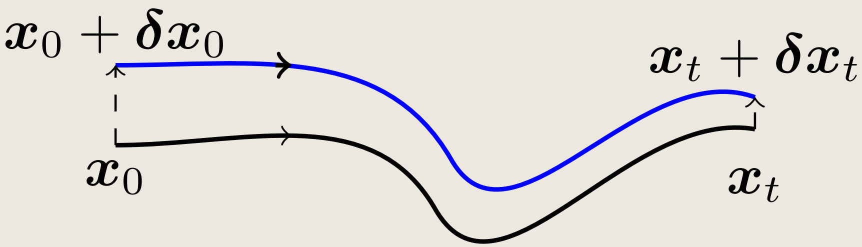


Variational Dynamics

$$\boldsymbol{\delta x}_{k+1} = \boldsymbol{\Phi}_0^k \boldsymbol{\delta x}_0$$
$$\boldsymbol{\delta y}_k = \boldsymbol{\Psi}_0^k \boldsymbol{\delta x}_0$$

Initial perturbation

Variational Measurement Mapping Function

$$\boldsymbol{\Psi}_0^k = \frac{\partial \boldsymbol{h}(\boldsymbol{x}_k)}{\partial \boldsymbol{x}_k} \boldsymbol{\Phi}_0^k$$

References

A. Iserles, A First Course in the numerical analysis of differential equations, 2nd ed. Cambridge University Press, 2009.

Variational Observability for Nonlinear Systems

Systems-theoretic observability quantification for nonlinear systems

1 Variational Dynamics

2 A New Gramian

References

New results toward system-theoretic observability quantification

Var-Gram

$$\boldsymbol{V}_o\left(\boldsymbol{x}_0\right) := \boldsymbol{\Psi}^\top \boldsymbol{\Psi} \in \mathbb{R}^{n_x \times n_x}$$

Obs-mat

$$\boldsymbol{\Psi} := \left[(\boldsymbol{\Psi}_0^0)^\top, (\boldsymbol{\Psi}_0^1)^\top, \ldots, (\boldsymbol{\Psi}_0^{\mathrm{N}-1})^\top\right]^\top \in \mathbb{R}^{\mathrm{N}n_y \times n_x}$$

Y. Kawano and J. M. Scherpen, "Empirical differential Gramians for nonlinear model reduction," Automatica, vol. 127, p. 109534, 2021.

S. Hanba, "On the "Uniform" Observability of Discrete-Time Nonlinear Systems," IEEE Transactions on Automatic Control, vol. 54, no. 8, pp. 1925-1928, 2009.

Variational Observability for Nonlinear Systems

Systems-theoretic observability quantification for nonlinear systems

1 Variational Dynamics

2 A New Gramian

3 Some Connections

Equivalence to linear Gramian (Kalman1963)

Theoretical results 1 & 2

Equivalence to Empr-Gram (Lall et al. 1999)

$$\boldsymbol{V}_o\left(\boldsymbol{x}_0\right) = \boldsymbol{W}\left(\boldsymbol{x}_0\right)$$

$$\boldsymbol{V}_o\left(\boldsymbol{x}_0\right) = \boldsymbol{W}^{\varepsilon}\left(\boldsymbol{x}_0\right)$$

Observation 2 – Faster that the Empr-Gram

Only one trajectory is required as compared to $2n_x$ times

Observation 3 – Observability Metrics

Different metrics that can help inform operational decisions

References

R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," Journal of the Society for Industrial and Applied Mathematics Series A Control, vol. 1, no. 2, pp. 152-192, 1963.
S. Lall, J. E. Marsden, and S. Glavaški, "Empirical model reduction of controlled nonlinear systems," IFAC Proceedings Volumes, vol. 32, no. 2 , pp. 2598-2603, 1999.

Variational Observability for Nonlinear Systems

Systems-theoretic observability quantification for nonlinear systems

1 Variational Dynamics

2 A New Gramian

3 Some Connections

Theoretical Result 1 – Relating Var-Gram to exponential stability

$$\log \det(\boldsymbol{V}_o(\boldsymbol{x}_0)) \equiv \alpha \sum_{i=1}^{n_x} \lambda_{L,i}$$

Enables connecting LEs which measure response to uncertain initial conditions to systems theoretic measures

Theoretical Result 2 – Local observability of nonlinear systems

$$\frac{1}{2k}\rho\left(\boldsymbol{V}_o\right) = \frac{1}{2k}\lambda_{\max}\left(\boldsymbol{V}_o\left(\boldsymbol{x}_0\right)\right) < 1$$

A uniform local observability condition for nonlinear systems based on local observability and distinguishability

References

M. H. Kazma and A. F. Taha, "Observability for Nonlinear Systems: Connecting Variational Dynamics, Lyapunov Exponents, and Empirical Gramians," 2024. arXiv:2402.14711.
S. Hanba, "On the "Uniform" Observability of Discrete-Time Nonlinear Systems," IEEE Transactions on Automatic Control, vol. 54, no. 8, pp. 1925-1928, 2009.

Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 **Submodular Maximization**

$$f^* = \underset{\mathcal{S}\subseteq\mathcal{V},\ \mathcal{S}\in\mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

One definition of Submodularity

For any, $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V}$

$$f(\mathcal{A} \cup \{s\}) - f(\mathcal{A}) \geq f(\mathcal{B} \cup \{s\}) - f(\mathcal{B})$$

References

L. Lovász, Submodular functions and convexity, pp. 235-257. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983.

Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 Submodular Maximization

2 Optimality Guarantees

References

Theoretical performance guarantee

$$f^*_{\mathcal{S}} \geq \left(1 - \frac{1}{e}\right) f^*$$

Greedy Algorithm: iteratively select r sensors using marginal gains

$$\mathcal{G}_j(a) = f(\mathcal{S}_j \cup \{a\}) - f(\mathcal{S}_j)$$

$f^*_{\mathcal{S}}$ is guaranteed to be at least 63% of the optimal value f^*.

Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 **Submodular Maximization**

2 **Optimality Guarantees**

3 **Submodularity of the Measures**

References

Theorem 4 – Var-Gram is modular set function

$$\boldsymbol{V}_o(\mathcal{S}) = \sum_{j\in\mathcal{S}} \left\{ \boldsymbol{\Phi}_0^{k\top} \boldsymbol{c}_j^\top \boldsymbol{c}_j \boldsymbol{\Phi}_0^k \right\}_{k=0}^{\mathrm{N}-1} = \sum_{j\in\mathcal{S}} \boldsymbol{V}_o(j)$$

The set $\mathcal{S} \subseteq \mathcal{V}$ encodes binary nodes selected $\gamma = \{\gamma_j\}_{j=1}^{n_y}$

Proves analogous properties to linear Gramian.

Theorem 5 – Observability measures are submodular

$$f(\mathcal{S}) = \begin{cases} \operatorname{trace}\big(\boldsymbol{V}_o(\cdot)\big) \\ \log\det\big(\boldsymbol{V}_o(\cdot)\big) \\ \operatorname{rank}\big(\boldsymbol{V}_o(\cdot)\big) \end{cases}$$

Enables scalable solution to the sensor allocation problem with performance guarantees

T. H. Summers, F. L. Cortesi, and J. Lygeros, "On Submodularity and Controllability in Complex Dynamical Networks," IEEE Transactions on Control of Network Systems, vol. 3, pp. 91-101, mar 2016.

Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 **Submodular Maximization**

2 **Optimality Guarantees**

3 **Submodularity of the Measures**

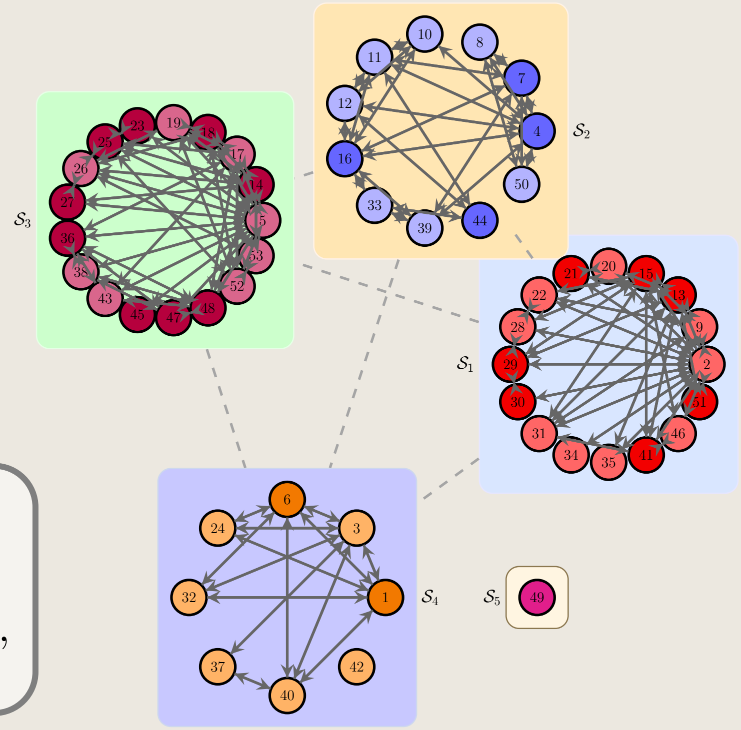
4 **Combustion Networks**

References

Two Combustion Networks

- $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$
- H_2O_2
  $n_r =$ 53 chemical reactions
  $n_x =$ 9 species model
- Methane CH_4
  $n_r =$ 325 chemical reactions
  $n_x =$ 53 species model

$$\sum_{i=1}^{n_x} q_{ji}\mathcal{R}_i \rightleftarrows \sum_{i=1}^{n_x} w_{ji}\mathcal{R}_i$$



Polynomial nonlinearities

$$q_j(\boldsymbol{x}) = d_j^{(f)} \prod_{i=1}^{n} x_i^{\alpha_{ji}} - d_j^{(b)} \prod_{i=1}^{n} x_i^{\beta_{kj}}, r = 1, 2, \ldots, n_r,$$

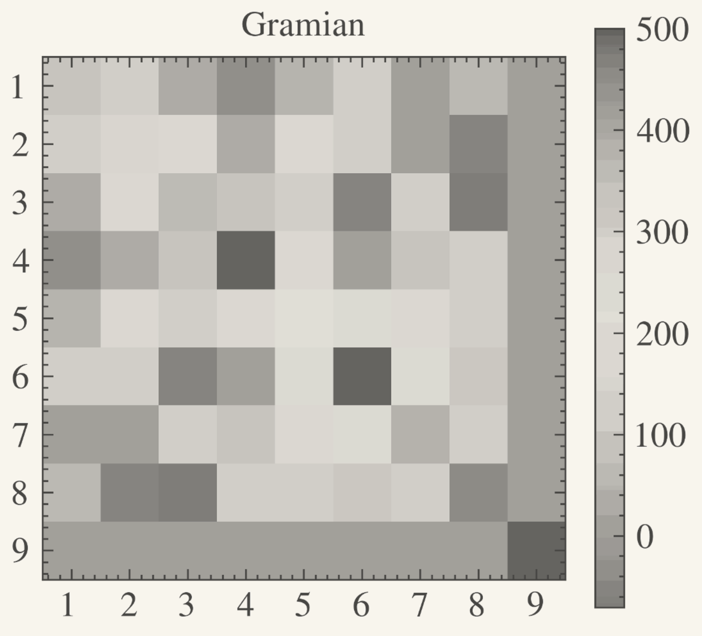
Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1. Submodular Maximization
2. Optimality Guarantees
3. Submodularity of the Measures
4. Combustion Networks

References

Computational Time (s)	
Var-Gram	Empr-Gram
0.0043	7.38
0.0032	8.53
0.489	115.05
0.467	112.06



The relevance

- Computational efficient; it extends well to large-scale systems
- New observability measures for nonlinear systems

S. Lall, J. E. Marsden, and S. Glavaški, "Empirical model reduction of controlled nonlinear systems," IFAC Proceedings Volumes, vol. 32, no. 2 , pp. 2598-2603, 1999.

Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 Submodular Maximization

2 Optimality Guarantees

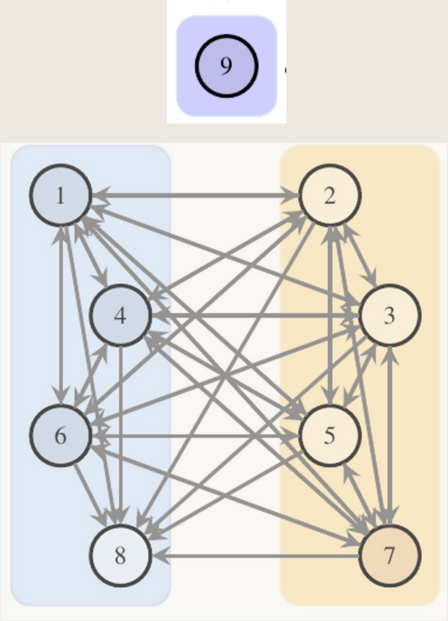
3 Submodularity of the Measures

4 Combustion Networks

References

Two Combustion Networks

- H_2O_2: 53 chemical reactions



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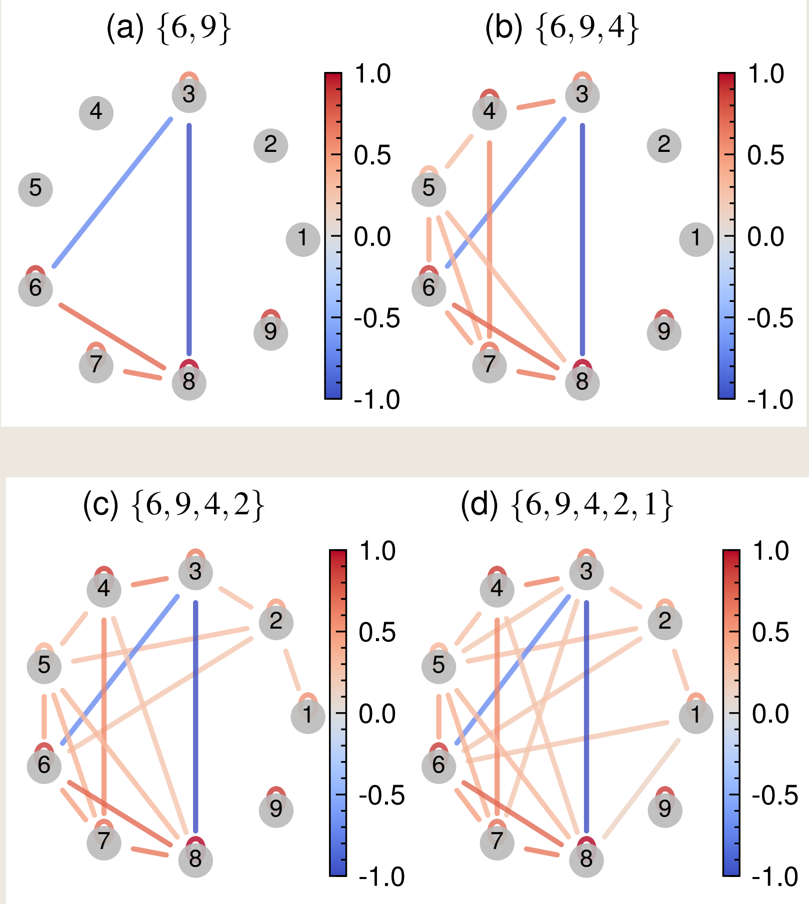


Observability-based Sensor Allocation Problem

A submodular optimization approach for sensor allocation using scalable observability measures

1 Submodular Maximization

2 Optimality Guarantees

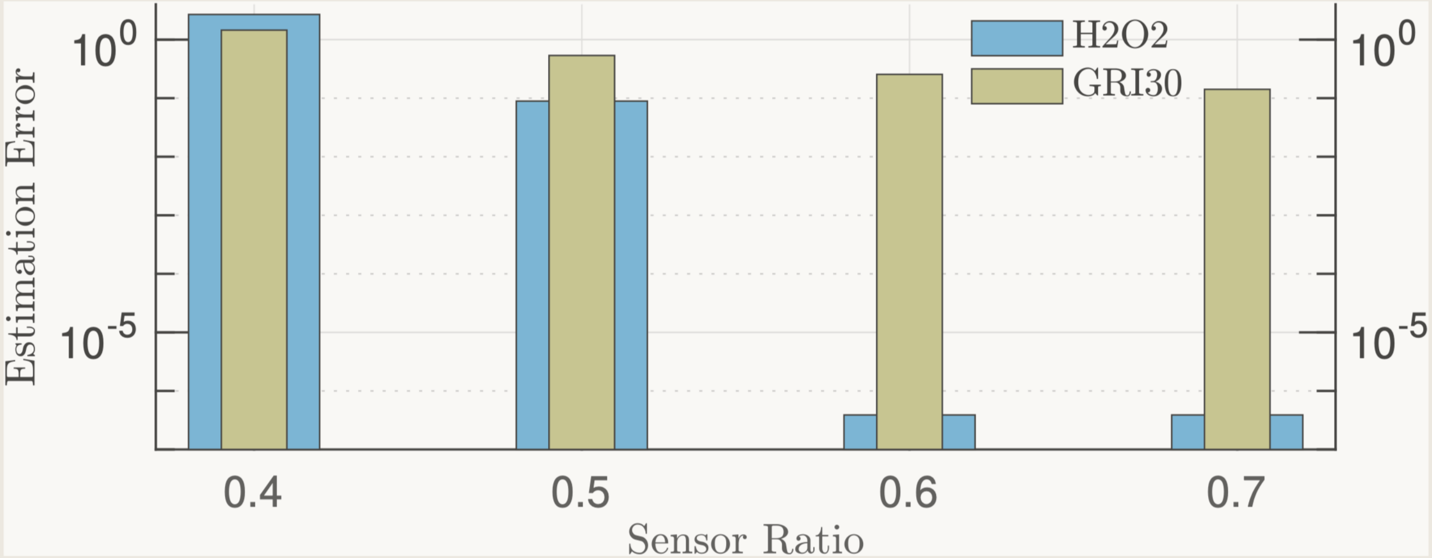
3 Submodularity of the Measures

4 Combustion Networks

References

The relevance

- *Optimality of the proposed sensor allocation framework*
- *Our ability to infer through limited sensing*



Revisiting Sensor Selection in Infrastructure Networks

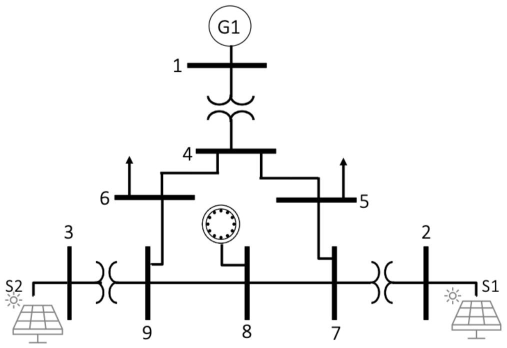
1 Power grids with renewables

2 Drinking water networks quality monitoring

Sensor Selection for Infrastructure Networks

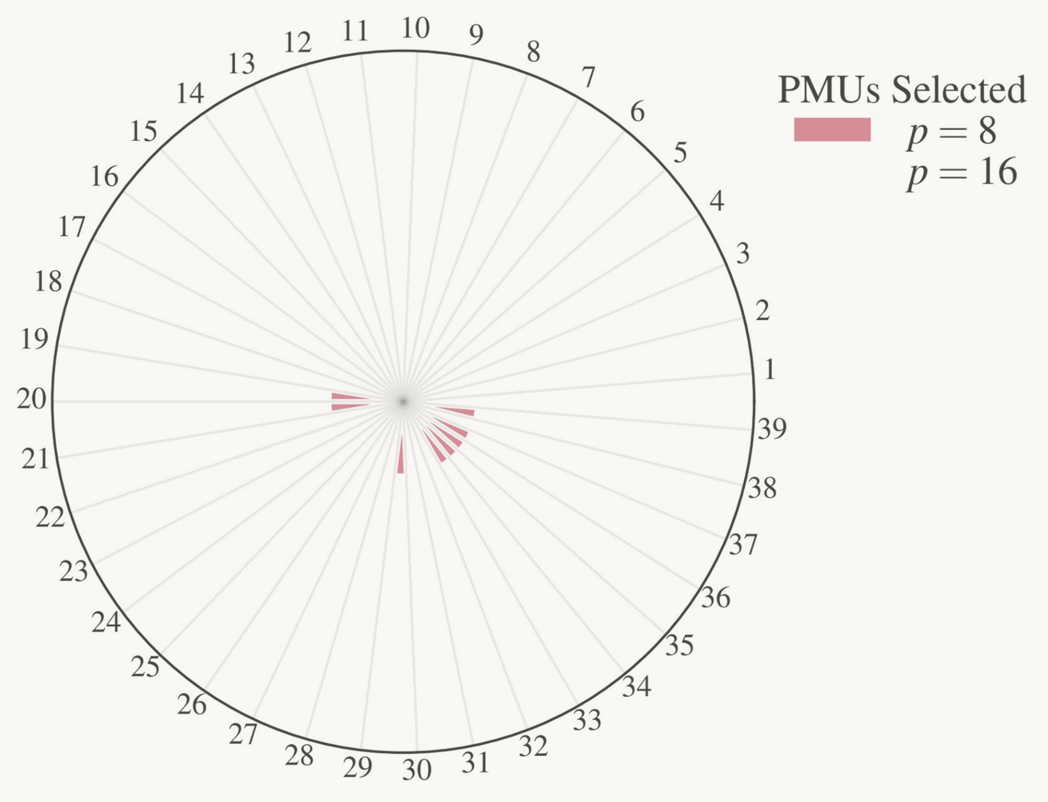
Robust systems-theoretic measures and sensor selection

1 On Power Grids



References

IEEE-case 39



The relevance

- *Modular Sensor selections* what is previously selected remains selected
- *Non-generator nodes* can be included as sensor locations
- *Variability of renewables* by accounting for changes in loads

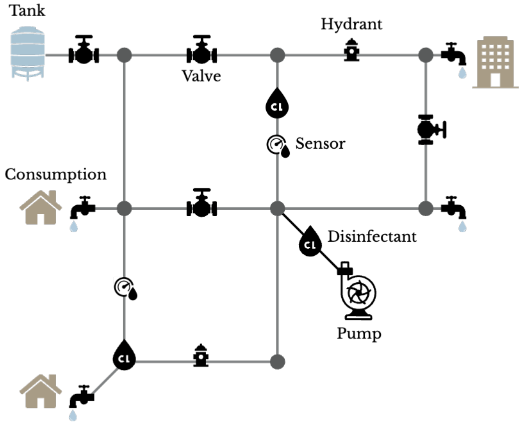
M. H. Kazma and A. F. Taha, "Revisiting the Optimal PMU Placement Problem in Multimachine Power Networks," IEEE Transactions on Control Systems Technology, vol. 33, no. 2, pp. 493-511, 2025.

Sensor Selection for Infrastructure Networks

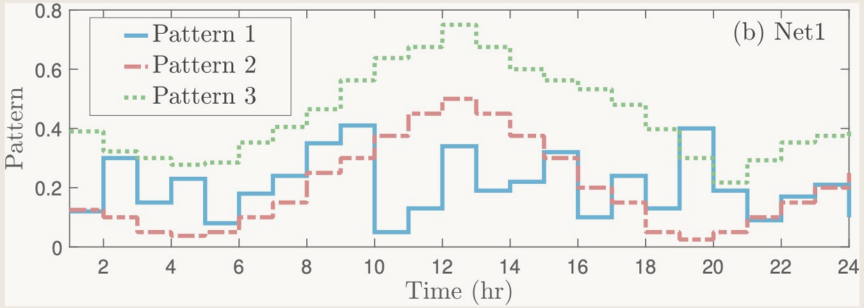
Robust systems-theoretic measures and sensor selection

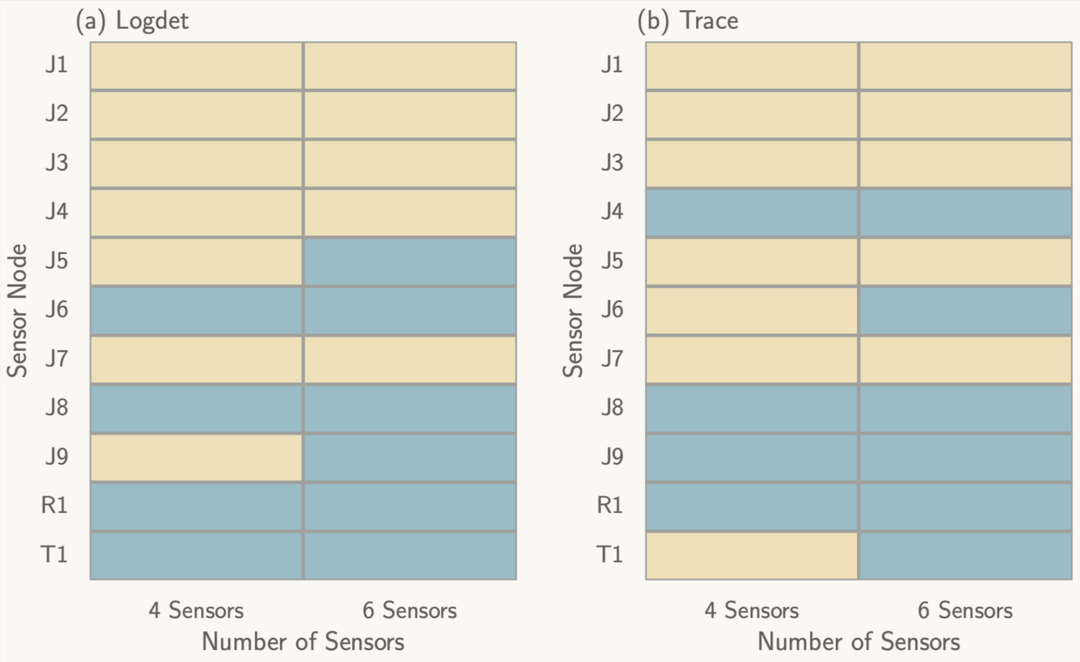
1 On Power Grids

2 On Drinking Water Networks



References





The relevance

- *Robust* to change in operating conditions; now only *one allocation is obtained*
- Enables *operational considerations;* for each network topology *different measures* result in different considerations

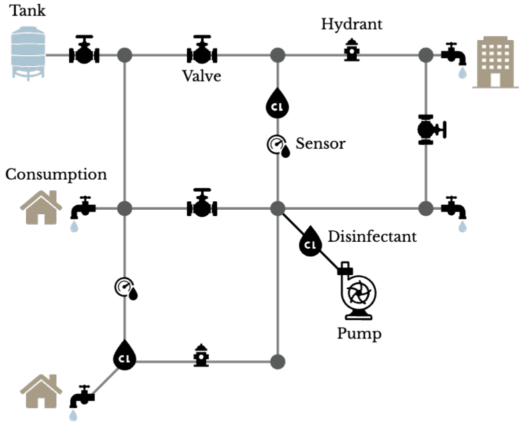
M. H. Kazma, S. M. Elsherif, and A. F. Taha, "Observability and generalized sensor placement for nonlinear quality models in drinking water networks, " Journal of Water Process Engineering, vol. 77, p. 108339, sep 2025.

Sensor Selection for Infrastructure Networks

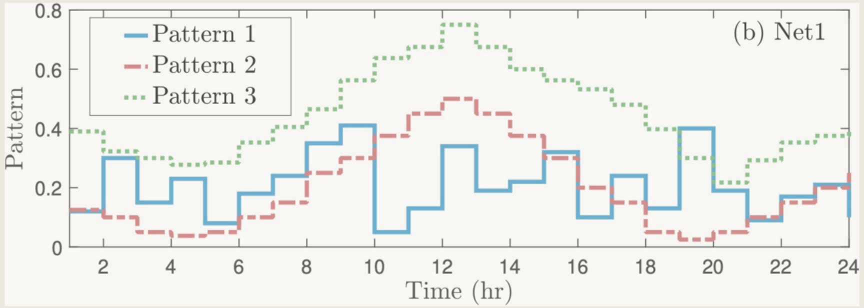
Robust systems-theoretic measures and sensor selection

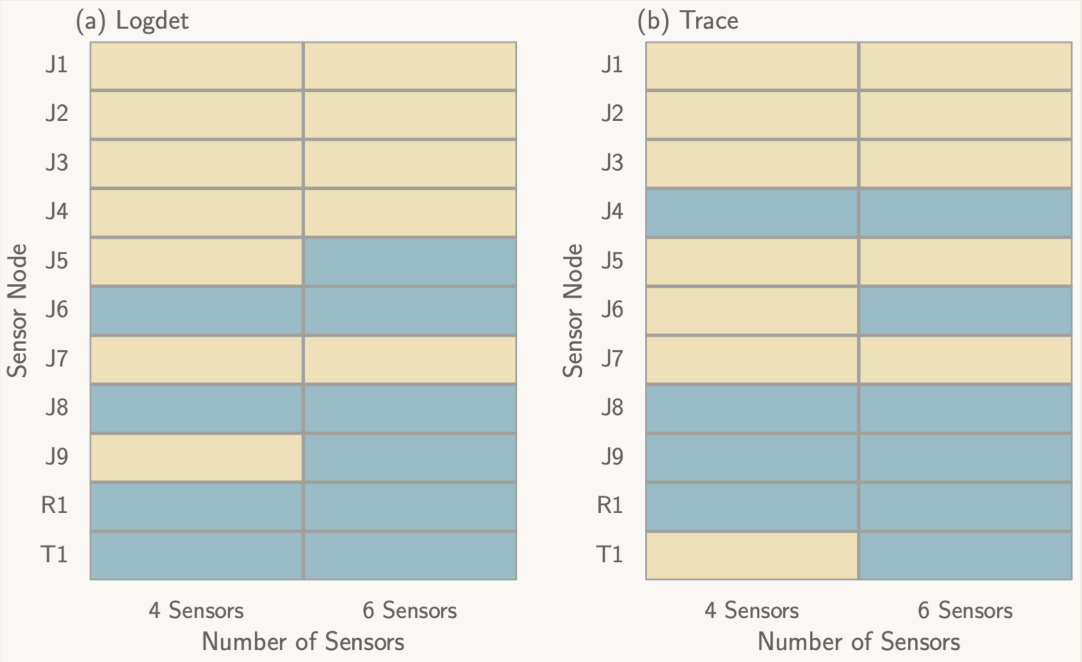
1 On Power Grids

2 On Drinking Water Networks



References





The relevance

Loop over demand profiles & WQ conditions: simulate WQ dynamics and compute Gramians $\mathbf{W}^{(\kappa)}(\mathcal{S}, \hat{\mathbf{x}}_0^{(\kappa)})$

↓

Greedy Algorithm: iteratively select r sensors using marginal gains

$$\mathcal{G}_j(a) = f(\mathcal{S}_j \cup \{a\}) - f(\mathcal{S}_j)$$

- Enables *operational considerations;* for each network topology *different measures* result in different considerations

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Partitioning of Complex Networks

Scalable partitioning via welfare maximization

5

Networks and their Partitions

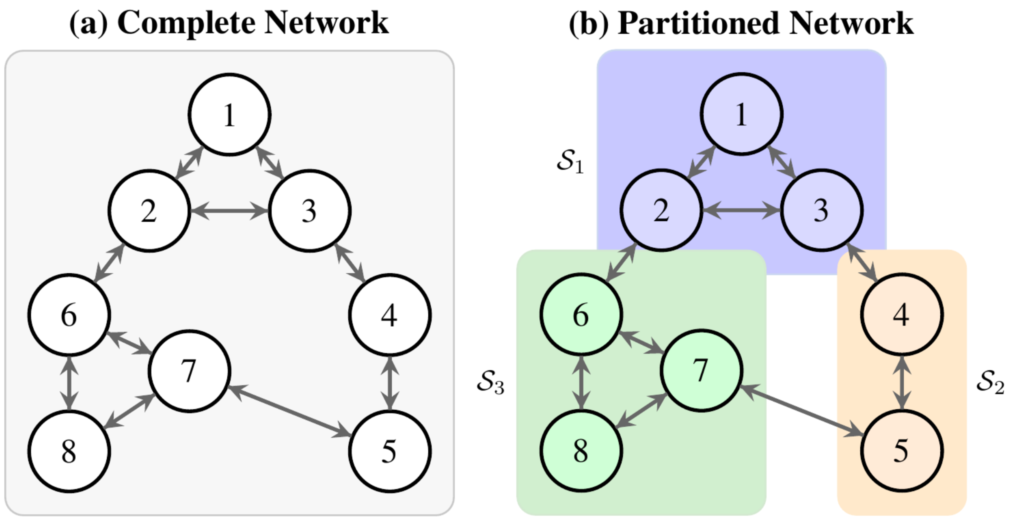
Enabling scalable solutions for large scale networks

Novel Clustering approach by maximizing subsystem welfare:
System-theoretic measures

Optimize graph-based objectives (spectral clustering and modularity maximization)
Does not account for the underlying dynamics

System measure guarantees
Does not guarantee bounds relative to global system behaviors

Combinatorial for Large-scale networks
Can be computational expensive to segment a large network



Networks and their Partitions

Enabling scalable solutions for large scale networks

1 **Submodular Maximization**

$$f^* = \underset{\mathcal{S}\subseteq\mathcal{V},\ \mathcal{S}\in\mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

Other Constraints that exhibit linear independence

Constraints that are inherent to real world problems

$$\mathcal{M} = (\mathcal{V}, \mathcal{I})$$
$$(i)\ \emptyset \in \mathcal{I}$$
$$(ii)\ A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$$
$$(iii)\ \mathcal{A}, \mathcal{B} \in \mathcal{I},\ |\mathcal{A}| < |\mathcal{B}|$$
$$\exists s \in \mathcal{B} \setminus \mathcal{A},\ \mathcal{A} \cup \{s\} \in \mathcal{I}$$

Budget constraint

$$\mathcal{I}_c = \{S \subseteq \mathcal{V} : |S| = r\}$$

(e.g., selecting r sensors from the entire network)

Partition constraint

$$\mathcal{V} = \bigcup_{i=1}^{\kappa} \mathcal{S}_i$$

$$\mathcal{I}_p = \{S \subseteq \mathcal{V} : |S \cap \mathcal{S}_i| \leq \kappa_i\}$$

(e.g., selecting sensors with at most κ_i per subsystem)

References

N. Rezazadeh and S. S. Kia, "Distributed Strategy Selection: A Submodular Set Function Maximization Approach," Automatica, vol. 153, p. 111000, 2021.

Networks and their Partitions

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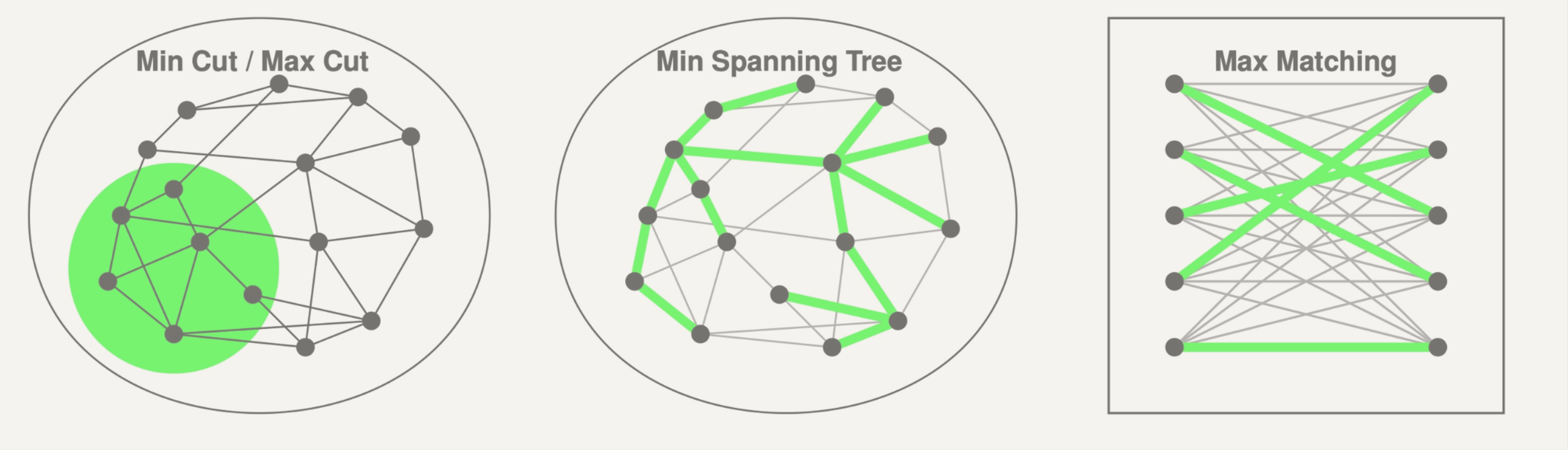

References

J. Vondrák, "Optimal approximation for the Submodular Welfare Problem in the value oracle model," Proceedings of the Annual ACM Symposium on Theory of Computing, pp. 67-74, 2008.

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$$f^* = \underset{\mathcal{S}\subseteq\mathcal{V},\ \mathcal{S}\in\mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

System-theoretic partitioning of complex networks

$$\max_{\mathcal{S}_1,\ldots,\mathcal{S}_\kappa} \quad f\left(\mathcal{S}_1,\ldots,\mathcal{S}_\kappa\right) = \sum_{i=1}^{\kappa} f_i\left(\mathcal{S}_i\right)$$

$$\text{s.t.} \quad \bigcup_{i=1}^{\kappa} \mathcal{S}_i = \mathcal{V},\ \mathcal{S}_i \cap \mathcal{S}_j = \emptyset \ \ \forall\, i \neq j$$

References

J. Vondrák, "Optimal approximation for the Submodular Welfare Problem in the value oracle model," Proceedings of the Annual ACM Symposium on Theory of Computing, pp. 67-74, 2008.

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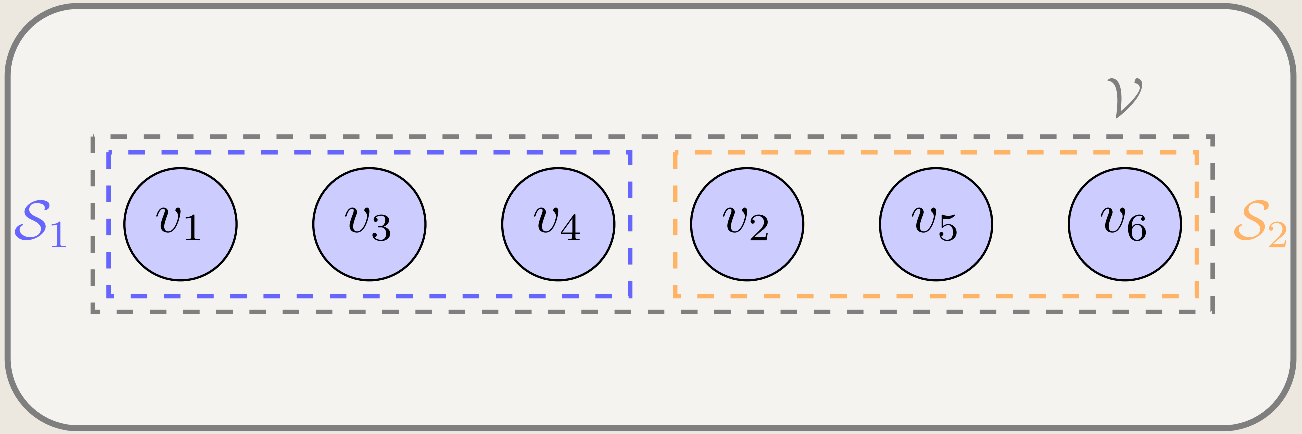

System-theoretic partitioning of complex networks

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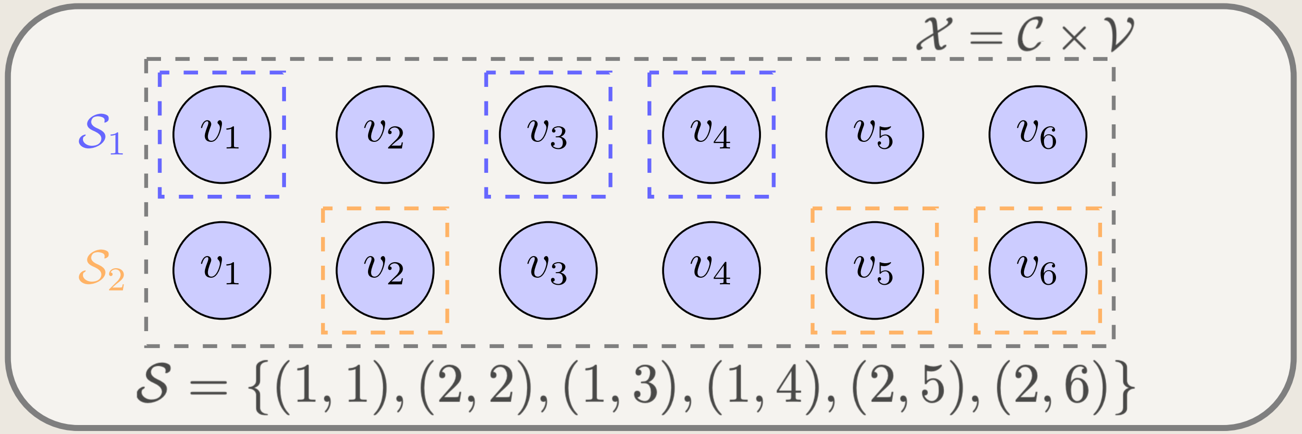
Networks and their Partitions

Enabling scalable solutions for large scale networks

1 **Submodular Maximization**

$\mathcal{S} \subseteq \mathcal{X},\ |\mathcal{S} \cap (\mathcal{C} \times \{v\})| \leq 1$



System-theoretic partitioning of complex networks

$$\max \ f(\mathcal{S}) = \sum_{i \in \mathcal{C}} f_i\Big(\{v \in \mathcal{V} \mid (i,v) \in S\}\Big)$$

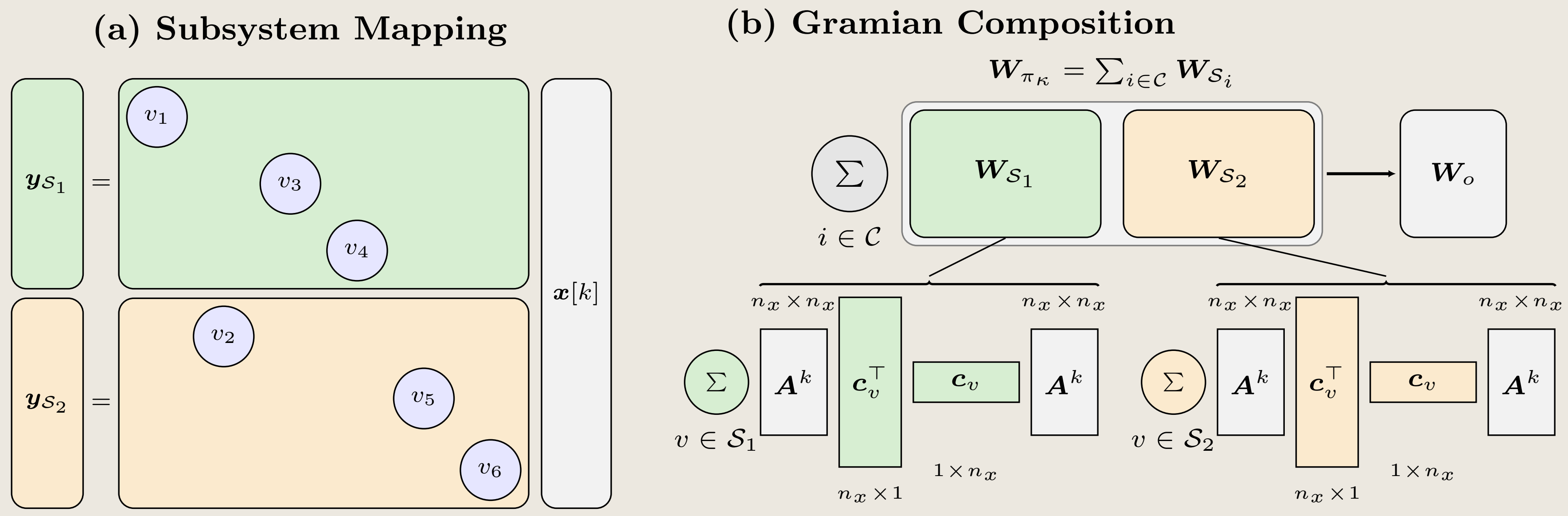
$$\text{s.t. } \{\mathcal{S} \subseteq \mathcal{X} : |\mathcal{S} \cap (\mathcal{C} \times \{v\})| \leq 1\}$$

References

M. H. Kazma and A. F. Taha, "Partitioning and Observability in Linear Systems via Submodular Optimization," IEEE Transactions on Automatic Control, no. i, pp. 1-16, 2026.

Networks and their Partitions

Enabling scalable solutions for large scale networks



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Networks and their Partitions

Enabling scalable solutions for large scale networks

1 Submodular Maximization

2 Subsystem Mapping

3 Multilinear Extension

$$F_{\mathcal{S}}^* = \underset{\mathcal{S}\subseteq\mathcal{V},\ \mathcal{S}\in\mathcal{I}_c}{\text{maximize}} \quad F(\mathbf{x})$$

$F : [0,1]^n \to \mathbb{R}_{\geq 0}$

Inclusion probability

$$F(\mathbf{x}) = \sum_{\mathcal{S}\subset\mathcal{V}} f(\mathcal{S}) \prod_{s\in\mathcal{S}} [\mathbf{x}]_s \prod_{s\notin\mathcal{S}} (1-[\mathbf{x}]_s)$$

How to compute?

Exclusion probability

References

G. Calinescu, C. Chekuri, M. Pál, and J. Vondrák, "Maximizing a Monotone Submodular Function Subject to a Matroid Constraint," SIAM Journal on Computing, vol. 40, pp. 1740-1766, jan 2011.

Networks and their Partitions

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$$F(\mathbf{x}) = \sum_{\mathcal{S} \subset \mathcal{V}} f(\mathcal{S}) \prod_{s \in \mathcal{S}} [\mathbf{x}]_s \prod_{s \notin \mathcal{S}} (1 - [\mathbf{x}]_s)$$

Method of expectation

$$F(\mathbf{x}) = \mathbb{E}\left[f\left(\mathcal{S}_{\mathbf{x}}\right)\right]$$

Element s inclusion and exclusion in $\mathcal{S}$

Other methods

- Exact Methods: closed form expressions
- Approx. Method: Taylor series expansion

References

G. Calinescu, C. Chekuri, M. Pál, and J. Vondrák, "Maximizing a Monotone Submodular Function Subject to a Matroid Constraint," SIAM Journal on Computing, vol. 40, pp. 1740-1766, jan 2011.

Networks and their Partitions

Enabling scalable solutions for large scale networks

1 Submodular Maximization

2 Subsystem Mapping

3 Multilinear Extension

4 Enabling Local Analysis

Context of node selection; sensor nodes

Sensor nodes $|\mathcal{R}| = r$

$$\max_{\mathcal{R}\subseteq\mathcal{V},\mathcal{R}\in\mathcal{I}_{pr_i}} f(\mathcal{R}) = f\left(\bigcup_{i\in\mathcal{C}}(\mathcal{R}\cap\mathcal{S}_i)\right)$$

Partition Matroid: $\{\mathcal{R}\subseteq\mathcal{V} : |\mathcal{R}\cap\mathcal{S}_i| \leq r_i, \forall i\in\mathcal{C}\}$

Theoretical Result – Bounds on subsystem observability measures

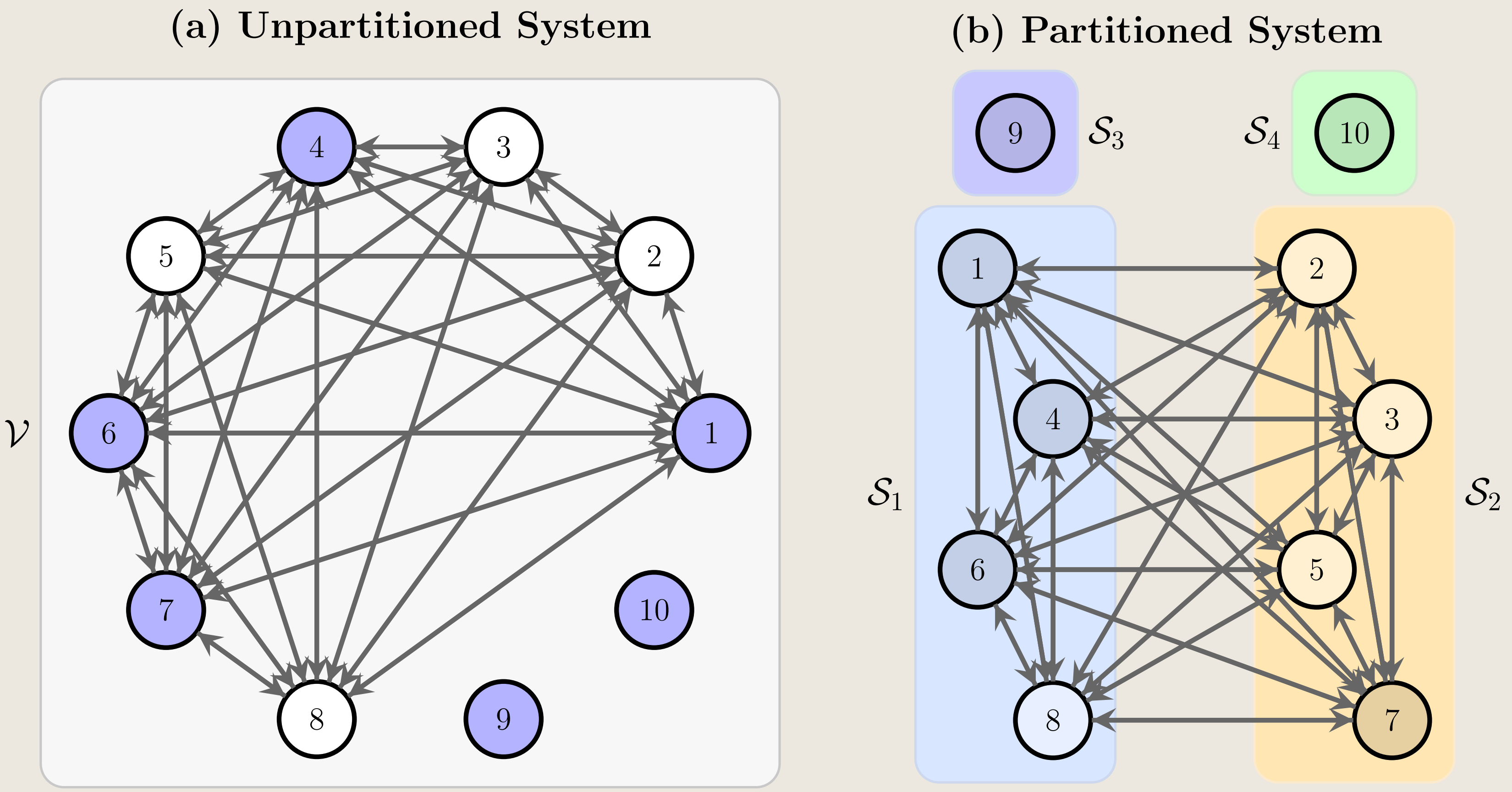
$$f\left(\bigcup_{i\in\mathcal{C}}(\mathcal{R}\cap\mathcal{S}_i)\right) \geq \sum_{i\in\mathcal{C}} f_i(\mathcal{R}\cap\mathcal{S}_i)$$

References

M. H. Kazma and A. F. Taha, "Partitioning and Observability in Linear Systems via Submodular Optimization," IEEE Transactions on Automatic Control, no. i, pp. 1-16, 2026.

Networks and their Partitions

Enabling scalable solutions for large scale networks



Networks and their Partitions

Enabling scalable solutions for large scale networks

1 Submodular Maximization

2 Subsystem Mapping

3 Multilinear Extension

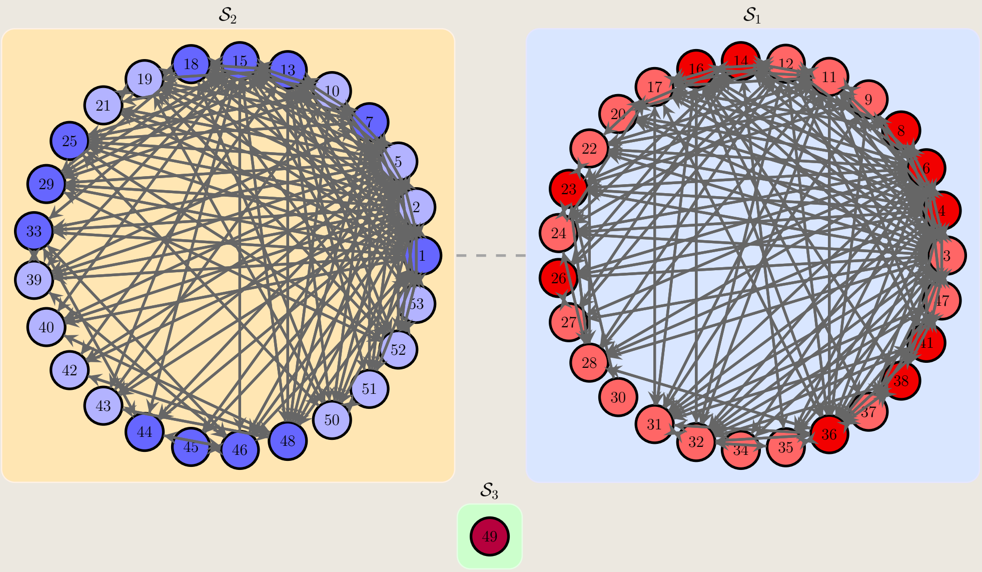
4 Enabling Local Analysis

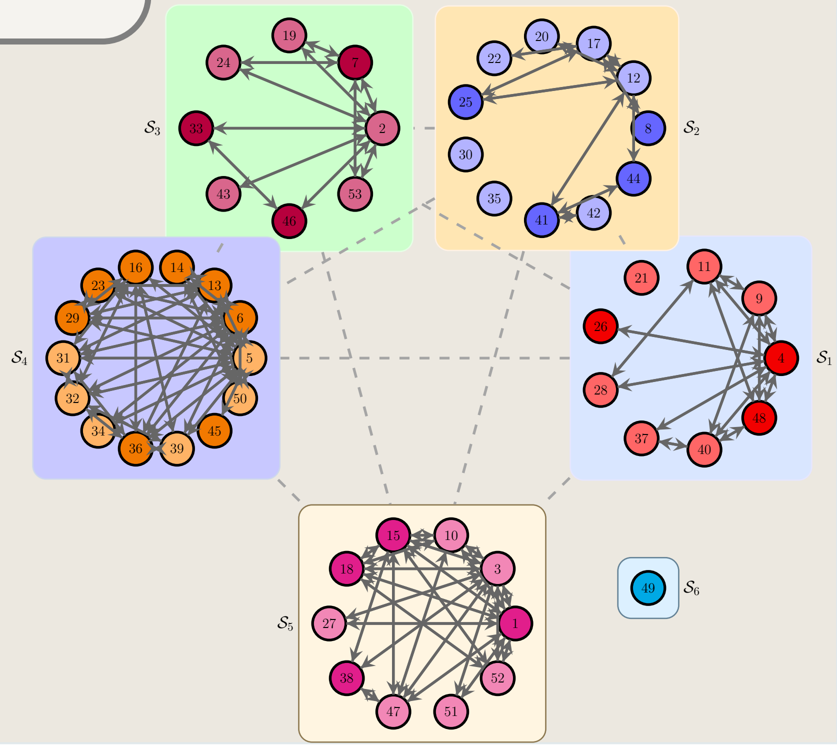
4 On Combustion Networks

References

The relevance

- Yields observable partitions while balancing network inter-intra connections
- Computational tractable at scale





Connections Between DPPs and Gramians in Control

6

Probabilistic node selection via determinantal point processes

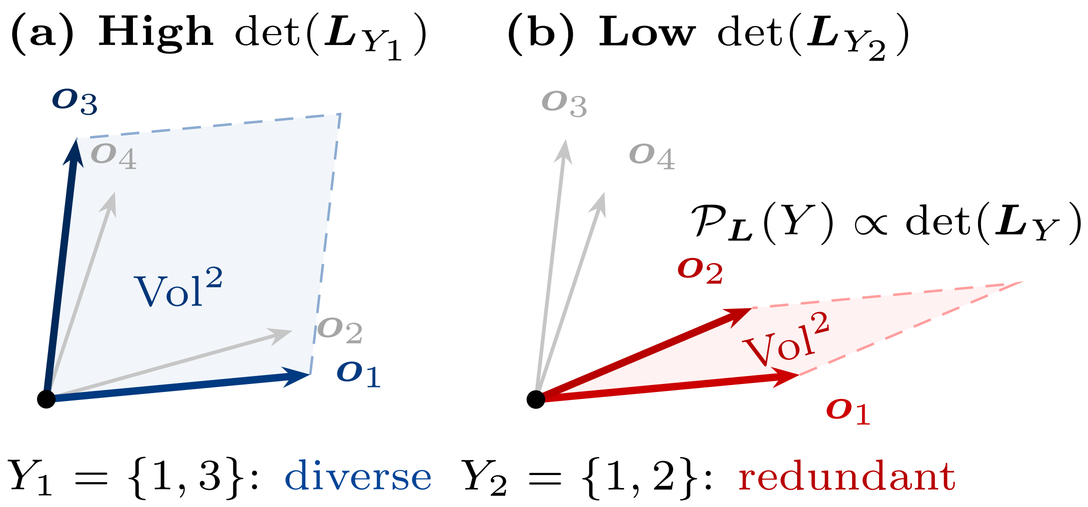
What are Determinantal Point Processes?

A probability model that naturally favors diversity and suppresses redundancy in subset selection

DPPs assign higher probability to subsets with elements complement one other

Widely used in machine learning, random matrix theory, and combinatorial optimization

- Given n items, a DPP picks a subset where similar items repel each other
- YouTube recommendations, Google search diversification, and document summarization
- Never before applied to control or dynamic systems



A Probabilistic Approach to Sensor Selection

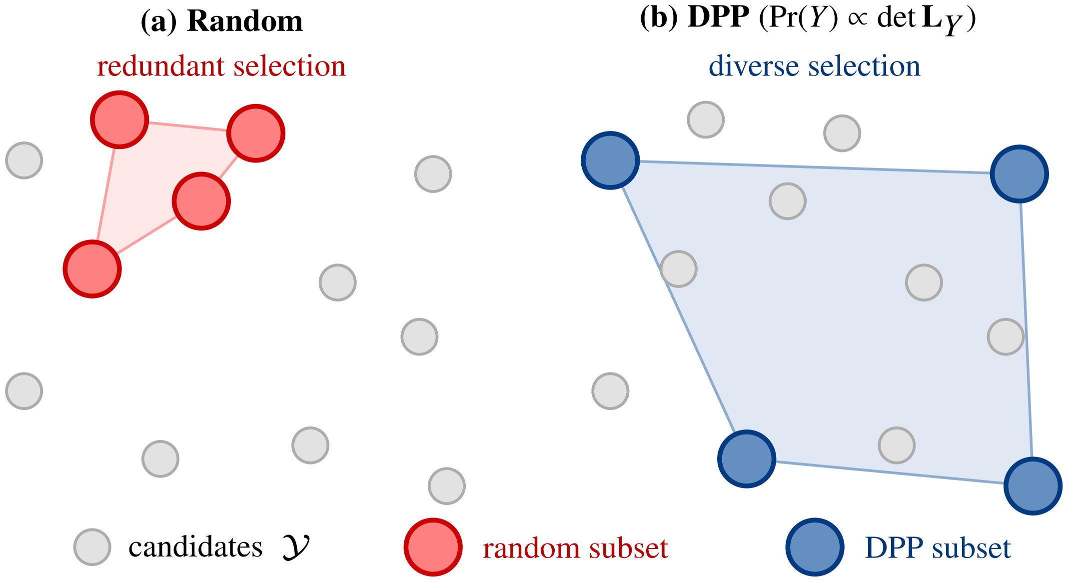
Establishing a connection to determinantal point processes (DPPs)

The observability Gramian is a determinantal point process — first connection between DPPs and Gramians in control

A new probabilistic perspective on node selection
Each sensor config Y gets probability P(Y) where diversity is favored, redundancy suppressed

k-DPP MAP recovers greedy logdet sensor selection
Sampling can yield near-optimal diverse configuration with 100% node coverage

New spectral insights not possible from a deterministic perspective
Effective observable rank and negative dependence



A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

1 Determinantal Processes

$$f^* = \underset{\mathcal{S}\subseteq\mathcal{V},\, \mathcal{S}\in\mathcal{I}}{\text{maximize}} \quad f(\mathcal{S})$$

Gramians as the DPP matrix induces a probability distribution

$$f^* = \max_{\mathcal{S}\subseteq\mathcal{V}, |\mathcal{S}|=r} \mathcal{P}^r_{\mathcal{S}}(\mathcal{S}) = \frac{\det(\boldsymbol{L}_S)}{\sum_{|S'|=r} \det(\boldsymbol{L}_{S'})}$$

$$\boldsymbol{V}_o(\mathcal{S}) = \boldsymbol{\Psi}(\mathcal{S})^\top \boldsymbol{\Psi}(\mathcal{S})$$

References

A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

1 Determinantal Processes

$$\mathcal{P}^r_{\mathcal{S}}(\mathcal{S}) = \frac{\det(\boldsymbol{L}_S)}{\sum_{|S'|=r} \det(\boldsymbol{L}_{S'})} \rightarrow \det(\boldsymbol{L} + \boldsymbol{I})$$

Gramians as the DPP matrix induces a probability distribution

$$\boldsymbol{L} = \boldsymbol{W}_o = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{S}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\det(\boldsymbol{L}_{\mathcal{S}})$	1	4	3	1	8	4	3	8
$\mathcal{P}_{\boldsymbol{L}}(\mathcal{S})$	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{3}{32}$	$\frac{8}{32}$

References

A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

1 Determinantal Processes

2 Some New Results

A marginal matrix based on the Gramian $\boldsymbol{K} = \boldsymbol{W}_o\left(\boldsymbol{W}_o + \boldsymbol{I}\right)^{-1}$

Theoretical Result 1 – Conditional Node Inclusion Probability

let $j \notin S$

Inclusion marginal probability

$$\mathbb{P}(j \in Y \mid S \subseteq Y) = \boldsymbol{K}_{jj} - \boldsymbol{K}_{j,S}\boldsymbol{K}_S^{-1}\boldsymbol{K}_{S,j}$$

where $\boldsymbol{K}_{j,S} \in \mathbb{R}^{1\times|S|}$ and $\boldsymbol{K}_S \in \mathbb{R}^{|S|\times|S|}$ are the submatrices of $\boldsymbol{K}$

For adaptive sensor scheduling, or expanding an existing sensor network when additional sensing resources become available

References

A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

1 Determinantal Processes

2 Some New Results

A marginal matrix based on the Gramian $\boldsymbol{K} = \boldsymbol{W}_o \left(\boldsymbol{W}_o + \boldsymbol{I}\right)^{-1}$

Theoretical Result 2 – Node Inclusion Monocity

Let $\boldsymbol{L}^{(1)} = \boldsymbol{W}_o^{(1)}$ and $\boldsymbol{L}^{(2)} = \boldsymbol{W}_o^{(2)}$. If $\boldsymbol{W}_o^{(1)} \preceq \boldsymbol{W}_o^{(2)}$, then

$$\mathbb{P}_{\boldsymbol{L}^{(1)}}(S \subseteq Y) \leq \mathbb{P}_{\boldsymbol{L}^{(2)}}(S \subseteq Y), \quad \forall S \subseteq \mathcal{Y},$$

where $\mathbb{P}_{\boldsymbol{L}^{(i)}}(S \subseteq Y) = \det\left(\boldsymbol{K}_S^{(i)}\right)$ is the probability that the configuration S.

How does modifying system parameters improve system-theoretic measures $(\boldsymbol{A}, \boldsymbol{C})$ and $(\boldsymbol{A}, \boldsymbol{B})$

References

A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

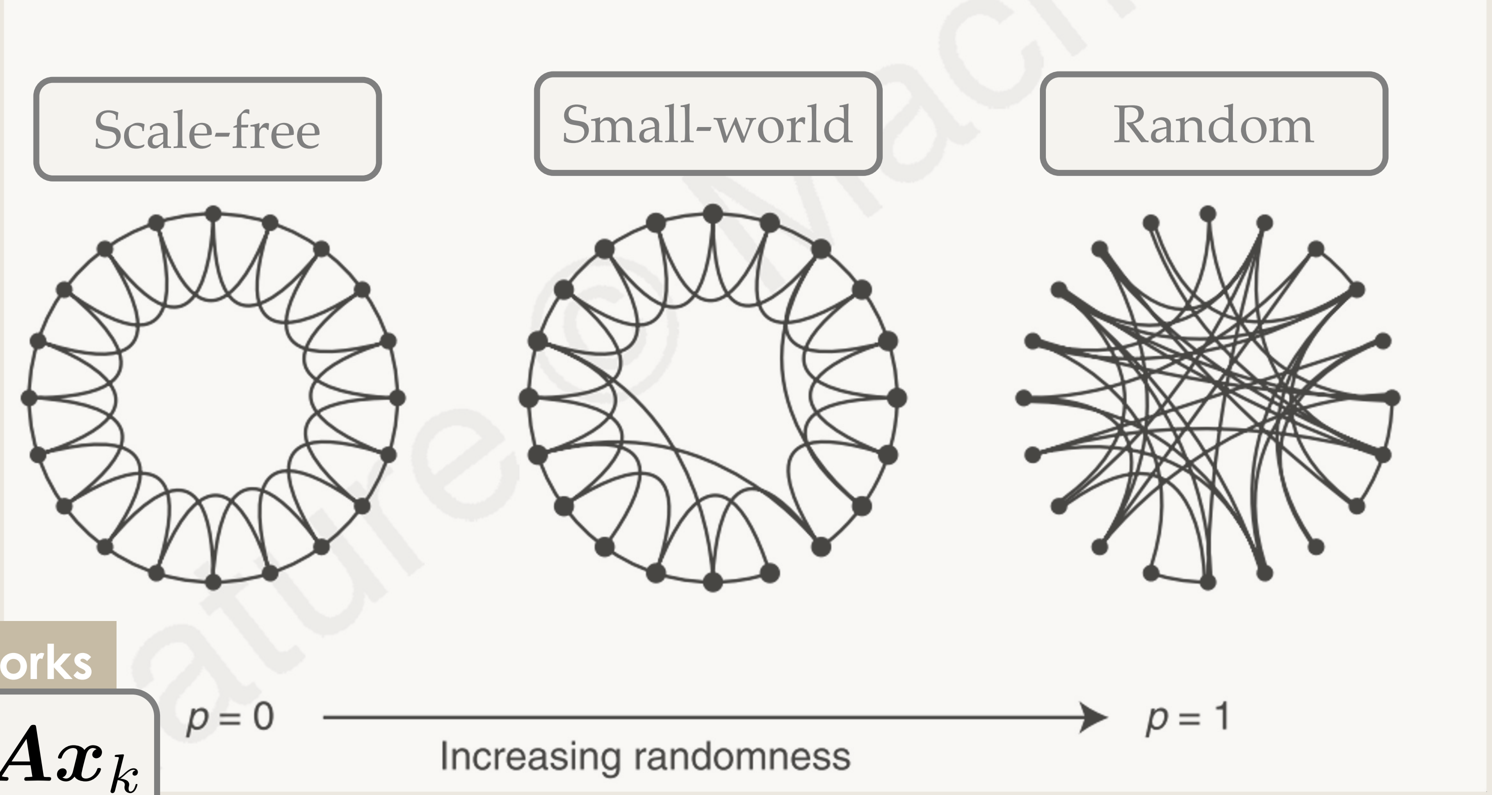
1. Determinantal Processes
2. Some New Results
3. Linear Networks Proof of Concept

References

Barabási–Albert (n = 200)

DWS (n = 122)

Erdős–Rényi (n = 200)



Linear Networks

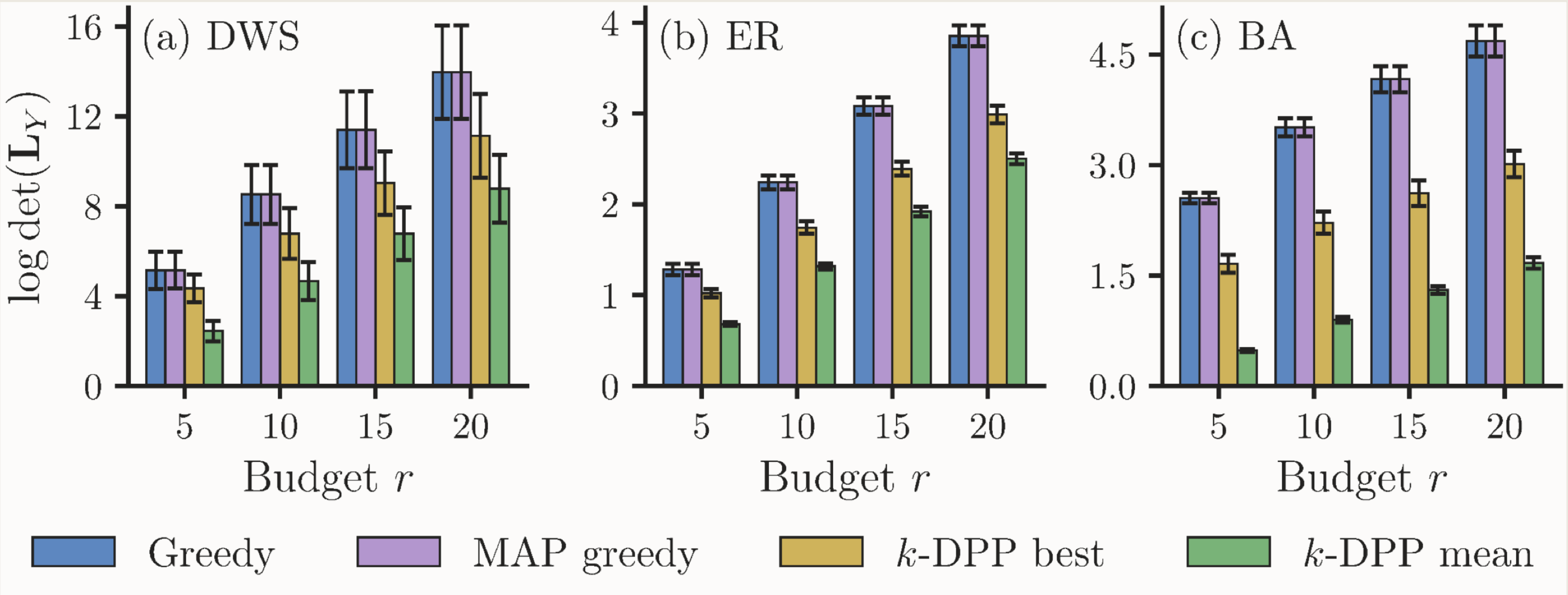
$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k$$

A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

The relevance

- MAP recovers classical greedy log det maximization
- Operators obtain an ensemble of diverse, near-optimal sensor configurations for robust monitoring under uncertainty

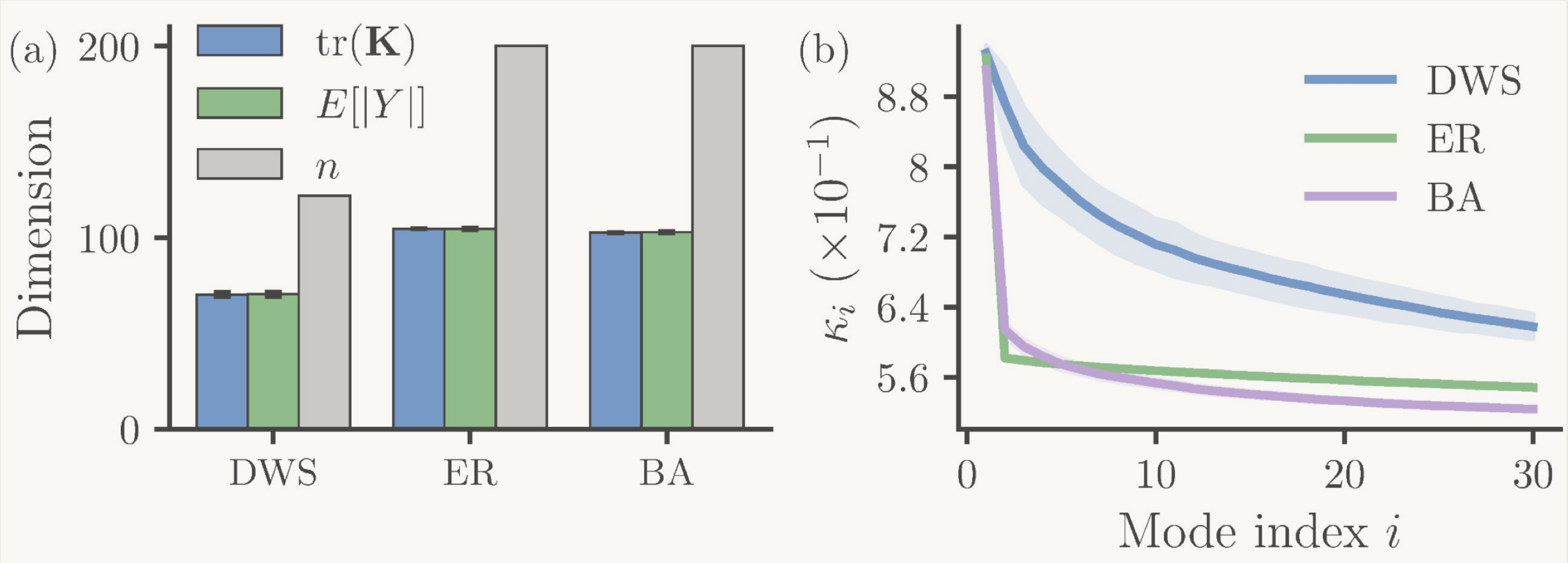


A Probabilistic Approach to Sensor Selection

Enabling new insights for node allocation in large-scale networks

The relevance

- Effective rank equivalence to trace; it informs on how many sensors the system might need for optimal monitoring
- More stable networks require more sensors to fully observe

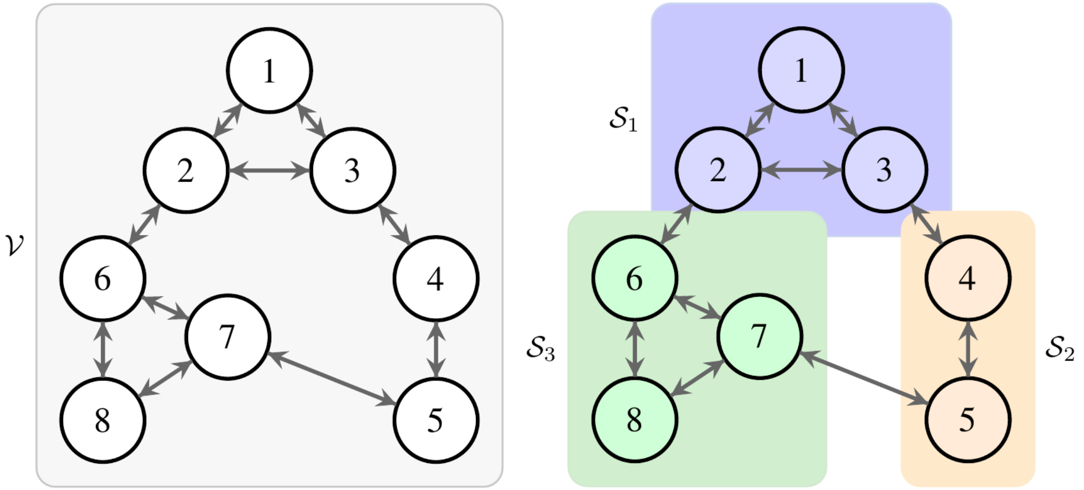


Future Work and the Broader Impact Perspective

Partitioning of networks in water and power networks

Sectoring the grid for stability
Inform operators on the extent to which variability affects the stability of connected subsystems

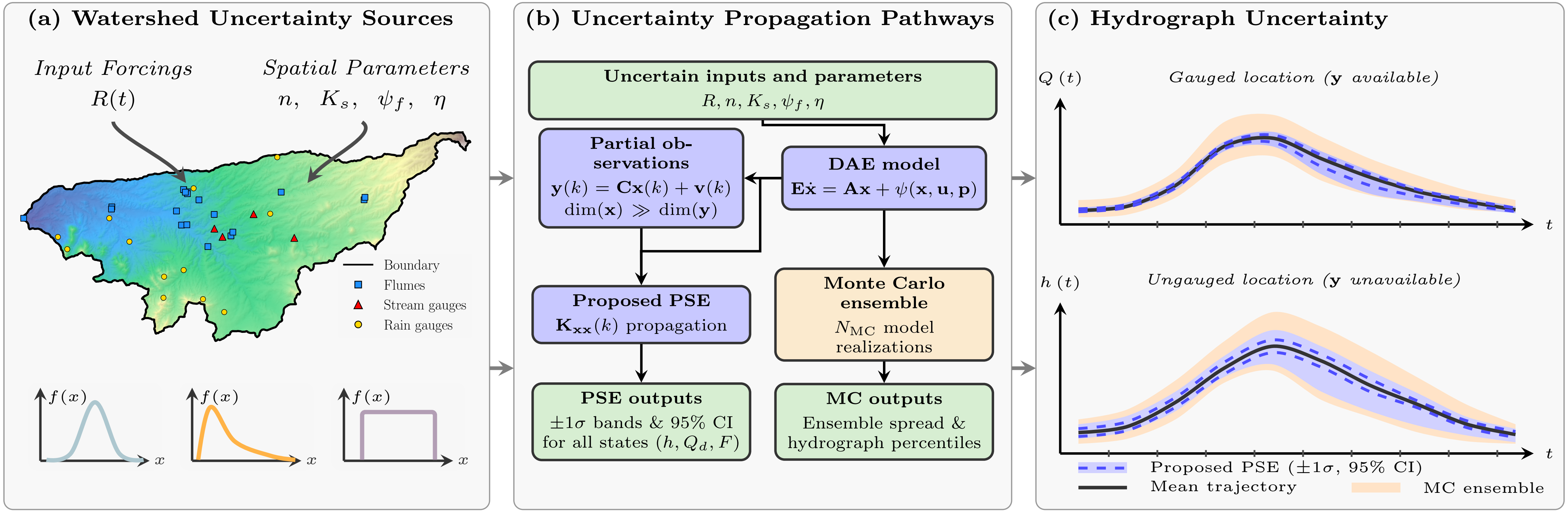
Zoning for disinfectant control
For maintaining safe water; network is segmented into pressure or quality zones to improve resilience to contamination events and to control DBP levels



Probabilistic node selection for control and monitoring in time-varying networks

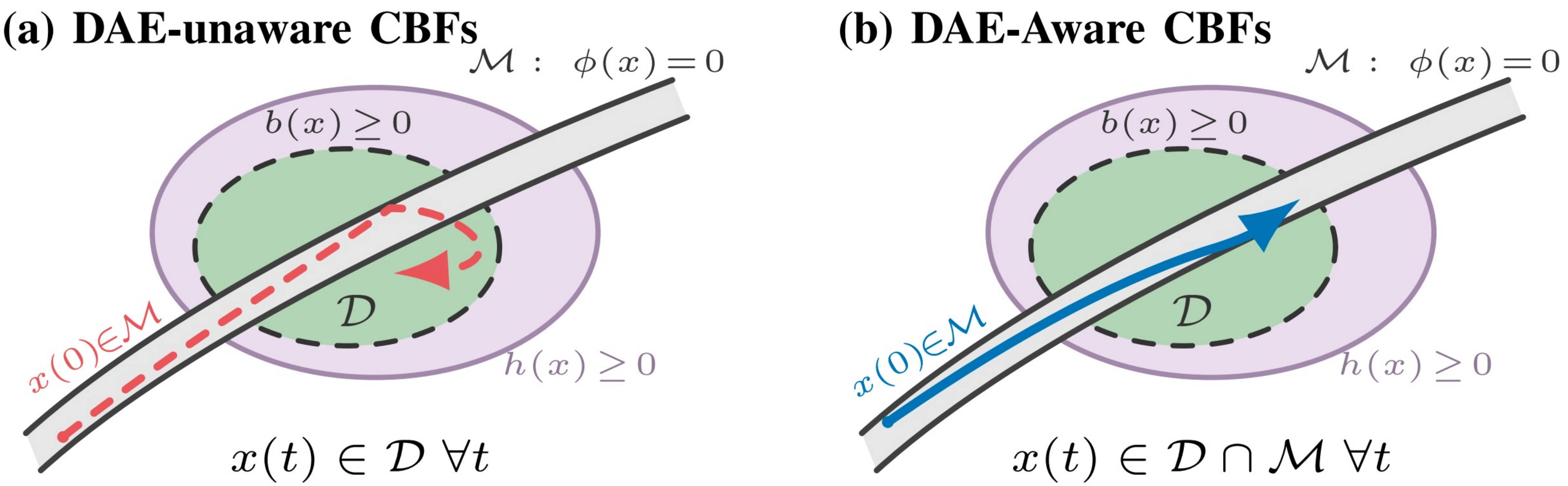
A new perspective on control
Practical notions such as confidence intervals for estimation quality, risk-aware monitoring priorities, anomaly detection, and reliability margins under changing operating conditions.

Future Work and the Broader Impact Perspective

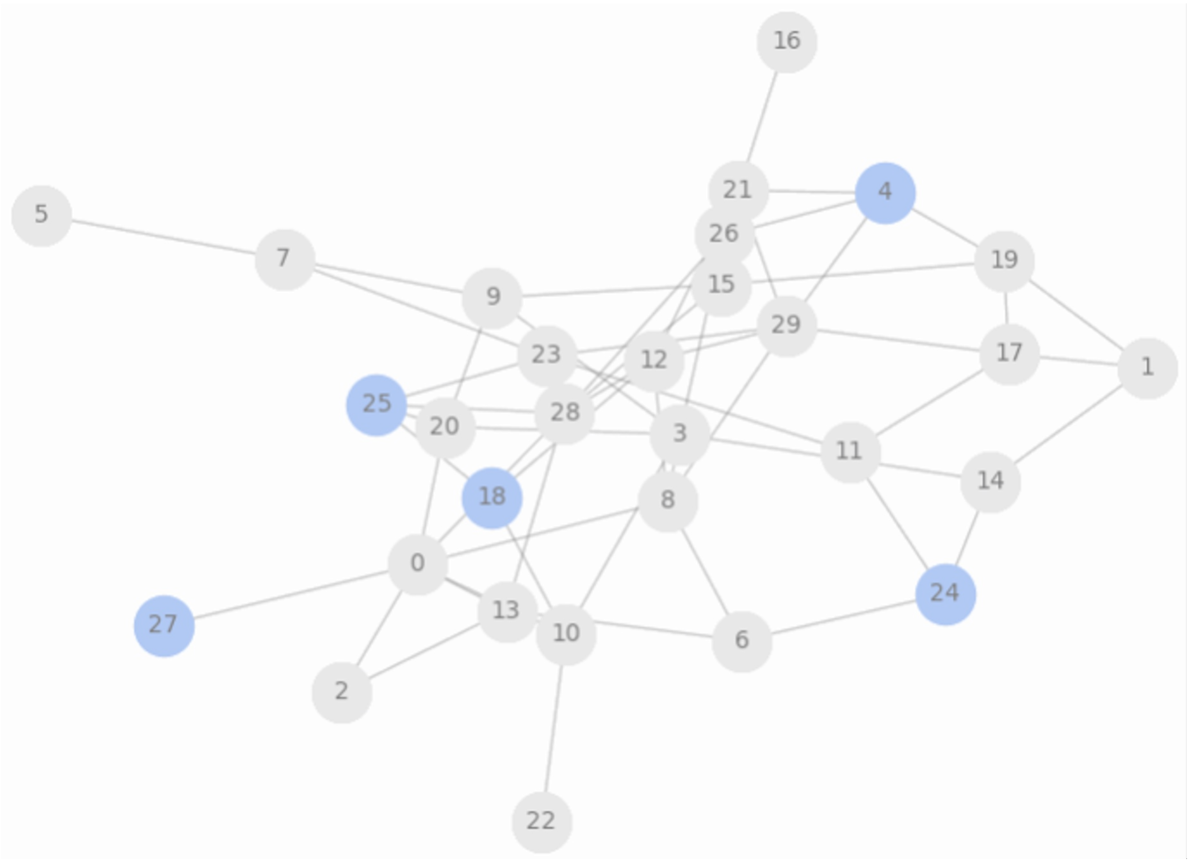


Distribution-agnostic state-space uncertainty propagation
Quantifying uncertainty in inputs and parameters through the DAE model formulation under partial measurements in natural dynamical systems.

Future Work and the Broader Impact Perspective



Extending safety to systems with physical constraints
Control barrier functions that take into account constraint manifolds along with the safety manifold imposed by physical constraints.



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Thank you @ KTH – Department of Decision and Control Systems on June 8th, 2026