VANDERBILT
School *of* Engineering

# CE 3300-01 – RISK, RELIABILITY, AND RESILIENCE ENGINEERING

## Continuous Random Variables

Book Reference: Chapters 3-4

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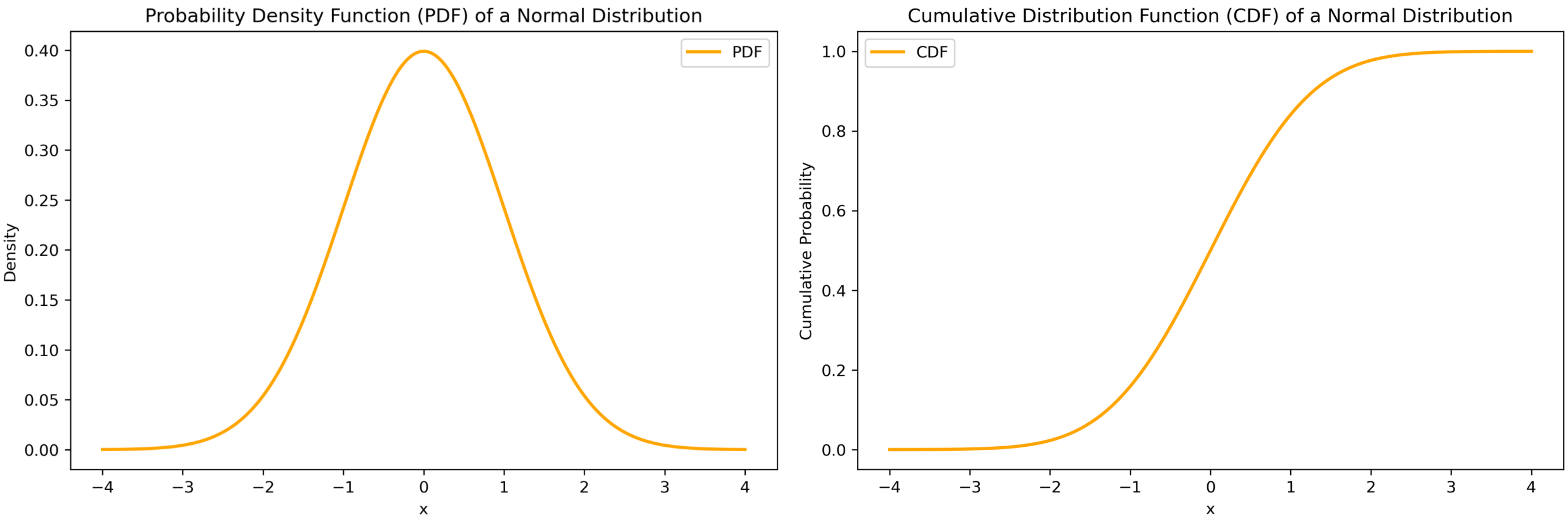
# Today: Continuous Random Variables



# Continuous RVs – Learning Objectives

- Continuous RVs and Probability Distribution
  - **Describe** continuous probability distributions
  - **Calculate** probability and moments of continuous probability distributions
  - **Distinguish** between types of probability distributions
  - **Solve** problems with both discrete and continuous R.V.

# Recall: Discrete Vs. Continuous RVs

- A RV is a variable whose value can take on one of several realizations such that each realization is associated with a likelihood of occurrence

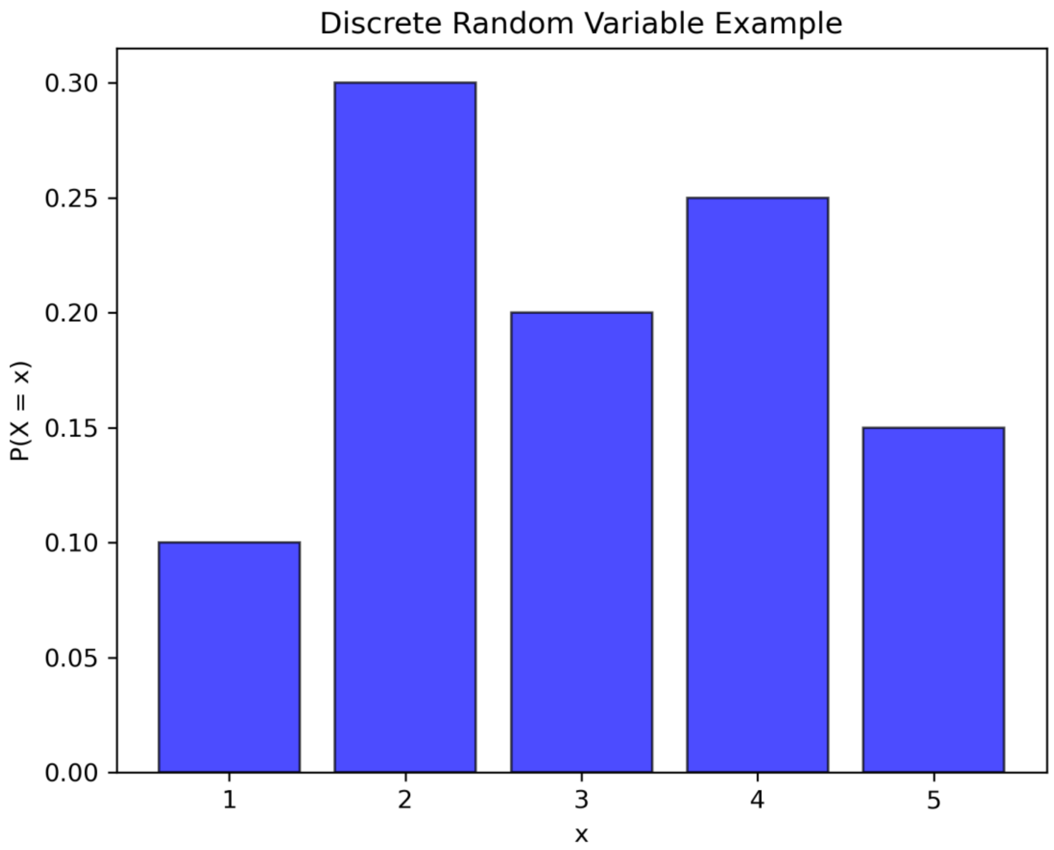
A **discrete** RV takes on a finite or countably infinite set of values. This means that the possible outcomes are distinct and can be listed individually.

A **continuous** RV can take on **any value** within an interval. Since there are infinitely many possible values, the probability of a single exact outcome is **zero.**

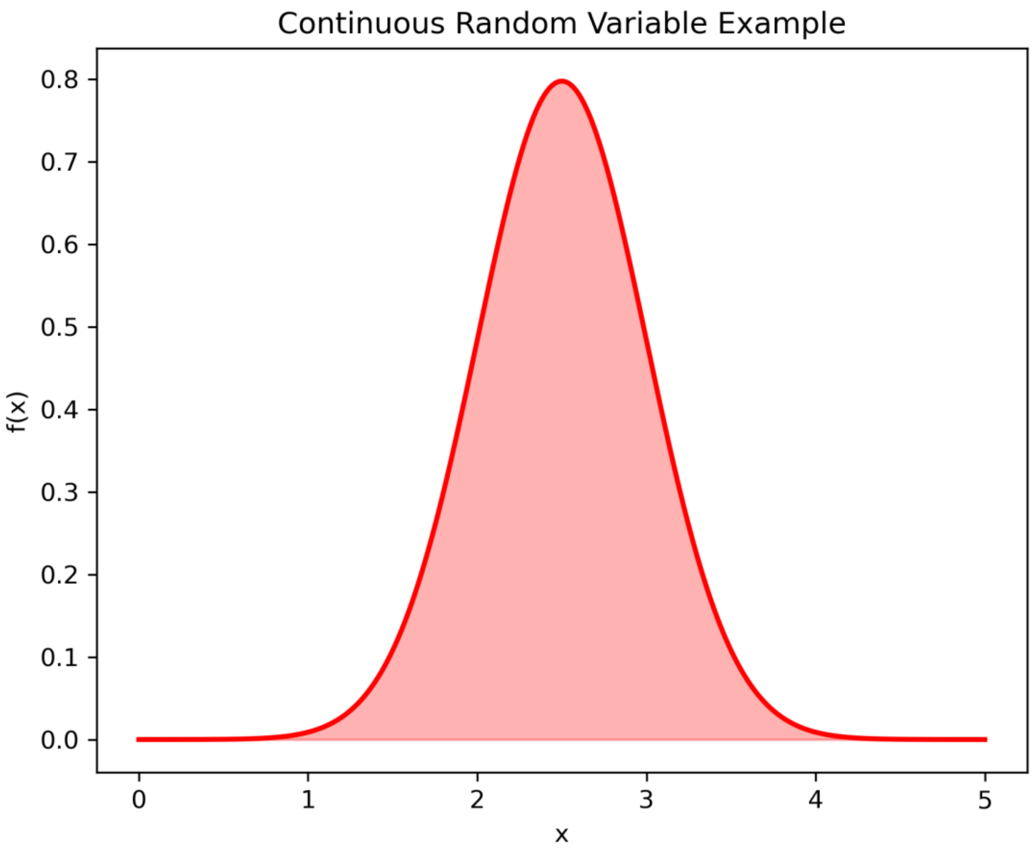
Probability is given by the probability mass function (PMF): $p_X(x)$ **discrete** RV

Probability is described by the probability density function (PDF): $f_X(x)$ **Continuous** RV

# Continuous Probability Distributions




$$P(X = x)$$




$$P(a < X \leq b) = \int_a^b f_X(x)dx$$

# Probability Density Function – PDF

## Discrete RV

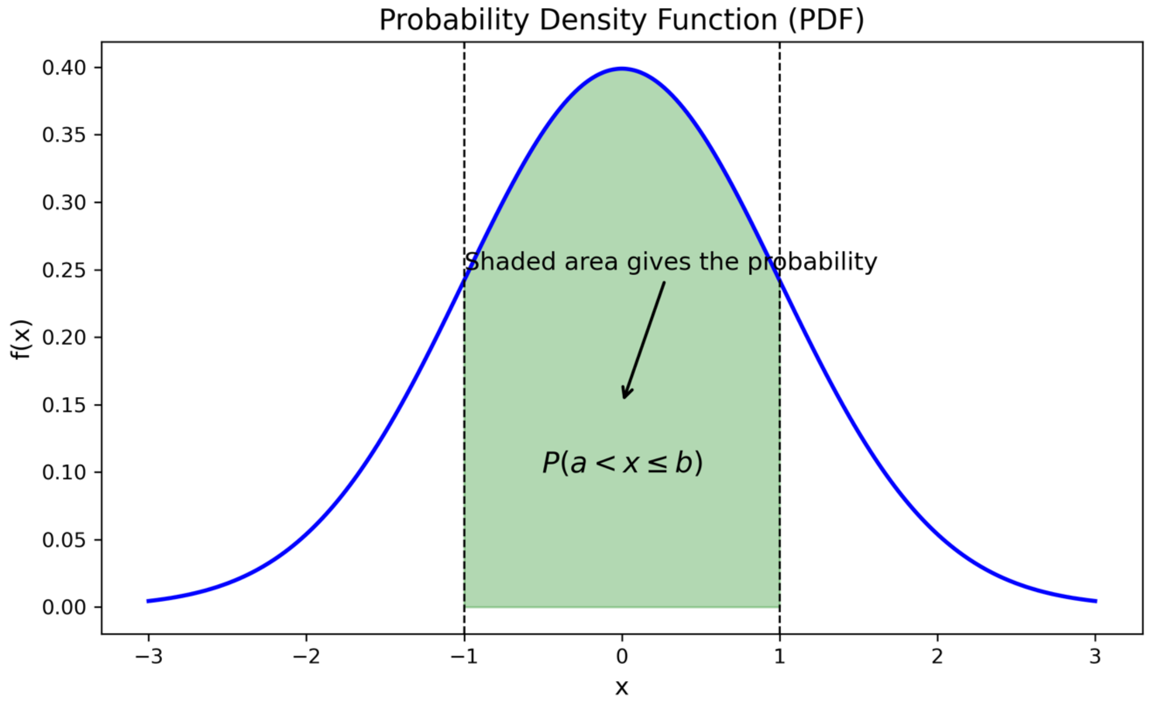
- Probability is given by the probability mass function (PMF)

$$f_x(x) = p_X(x_i) = \mathbf{P}(X = x_i)$$

## Continuous RV

- $f(x)$ is the probability density (distribution) function (PDF)
- The probability of the random variable taking a range of values is given by the area under the pdf

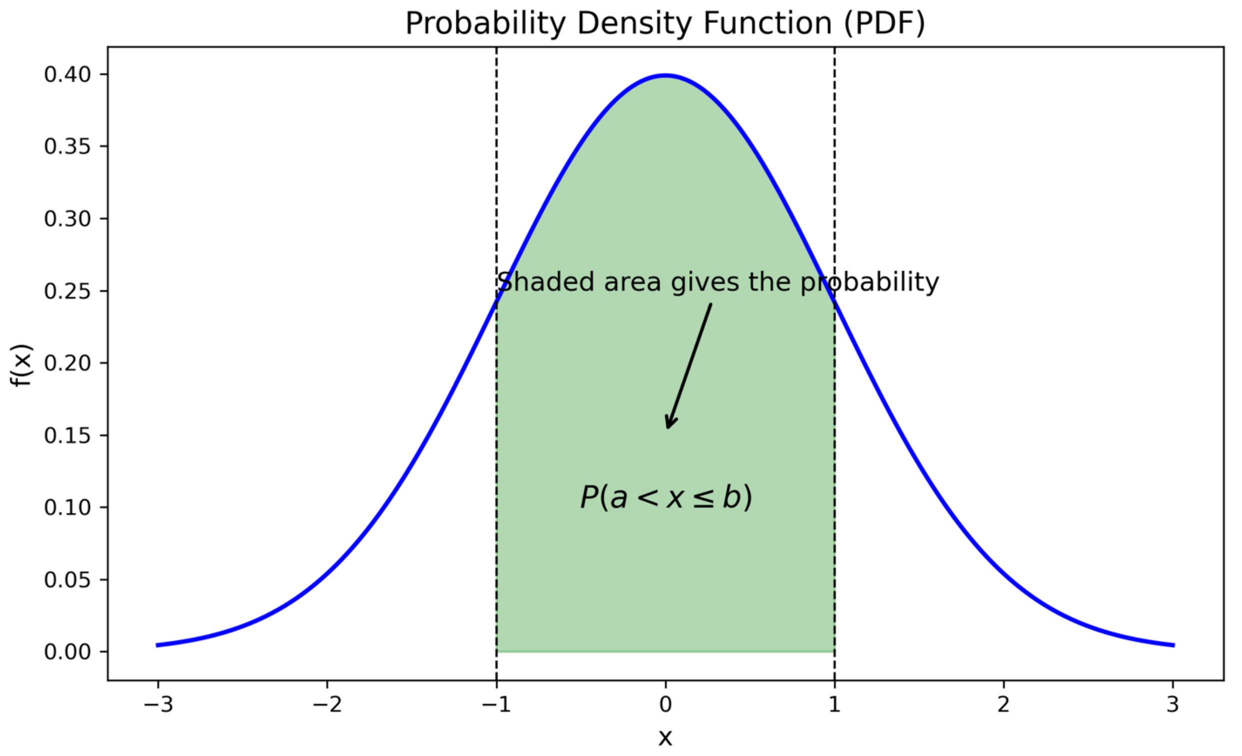
$$P(a < X \leq b) = \int_a^b f_X(x)dx$$



# Note the Equivalence

$$\mathbf{P}(X = a) = \int_a^a f_X(x)dx = 0$$

$$\mathbf{P}(a \le X \le b) = \mathbf{P}(a < X < b) = \mathbf{P}(a \le X < b) = \mathbf{P}(a < X \le b)$$



# Cumulative Distribution Function – CDF

## Discrete RV

- CDF

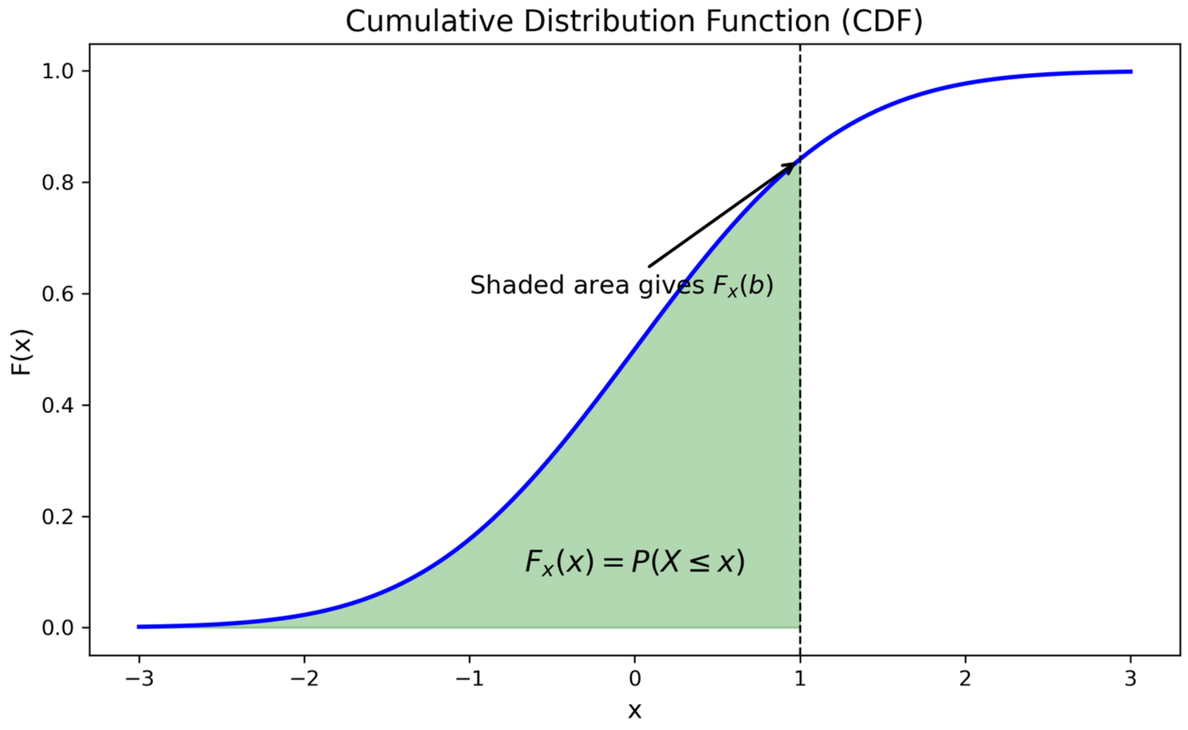
$$F_X(x) = \sum_{\text{all } x_i \le x} P(X = x_i) = \sum_{\text{all } x_i \le x} p_X(x_i)$$

## Continuous RV

- The CDF is obtained by integrating the PDF

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)dx$$

Or whatever the domain/lower bound is for the distribution




## Axioms

$$\sum_{x_i} p_X(x_i) = 1 \quad \int_{-\infty}^{+\infty} f_X(x)dx = 1$$

# Descriptors of a Continuous RV – Expected Value

The expected value represents the **average or mean** of the distribution.

Discrete RV

$$\mathbf{E}[X] = \sum_{x_i} x_i p_X(x_i)$$

Continuous RV

$$\mathrm{E}[\mathrm{X}] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

# Descriptors of a Continuous RV – Variance

The variance provides a measure of dispersion of X around its mean.
How widely or narrowly the values of the random variable are dispersed.

$$\mathrm{Var}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \mathbf{E}\left[X^2\right] - \mu_x^2$$

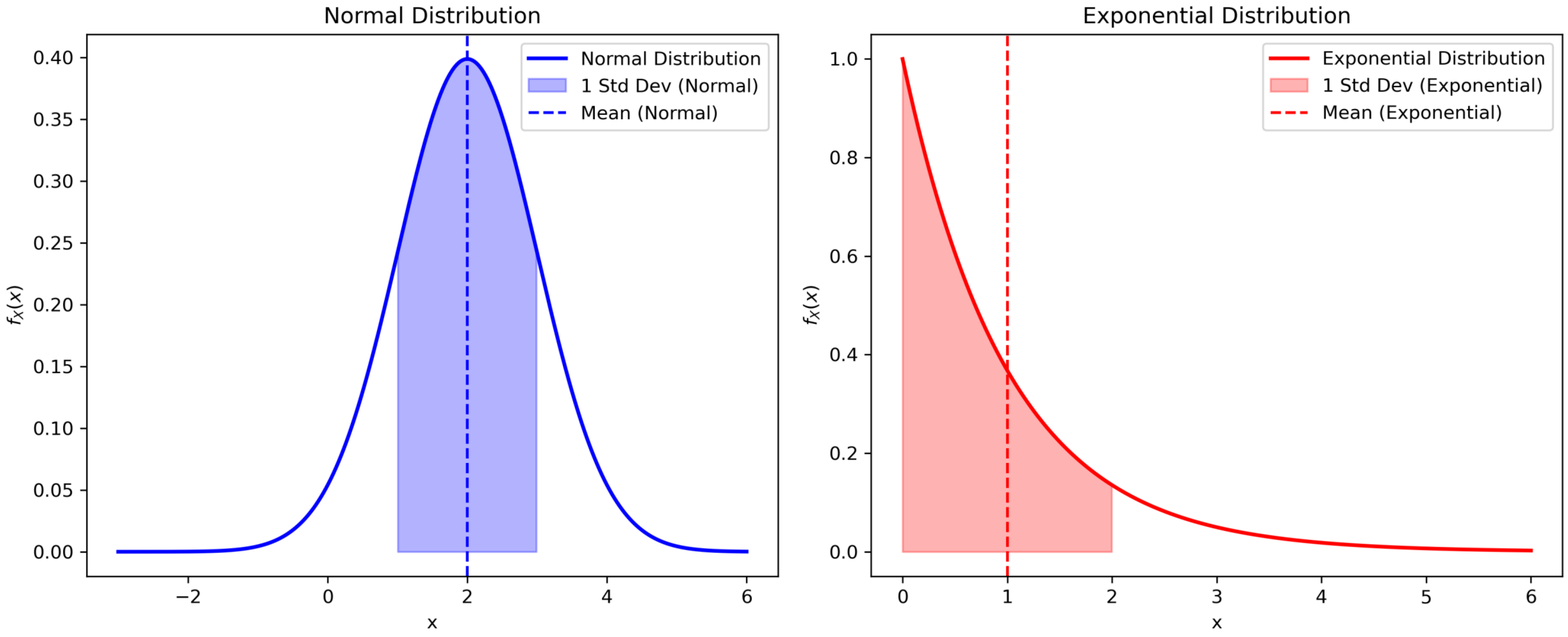
Discrete RV

$$\mathrm{Var}[X] = \sum x_i^2 p_X(x_i) - \left(\sum x_i p_X(x_i)\right)^2$$

Continuous RV

$$\mathrm{Var}[X] = \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_X(x)dx = \int_{-\infty}^{+\infty} x^2 f_X(x)dx - \left(\int_{-\infty}^{+\infty} x f_X(x)dx\right)^2$$

# Examples– Mean and Variance of a Continuous RV




$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

# Exponential Distribution - Definition

**Exponential Distribution** is a continuous probability distribution that models the **time between independent events** occurring at a constant average rate.

- The occurrence of an event constitutes a Poisson Process.
- It is often used to describe waiting times, such as time between customer arrivals or system failures. (**Recurrence time**)

**Real-world applications:**

- Time between arrivals of customers in a queue.
- Time until the failure of a machine component.
- Time between earthquakes in a given region.
- Time between the occurrence of floods

**Recall:**
Poisson distribution for "rate" or "# occurrences per unit"

# Exponential Distribution – Properties

The main property of the exponential distribution is the fact that it's **MEMORYLESS**

$$P(X \geq x_0 + x_1 \mid X \geq x_0) = P(X \geq x_1)$$

- Such a property is useful for calculations but may not represent reality in many applications

- Example: The probability of waiting additional time does not depend on how much time has already elapsed.

# Exponential Distribution – PDF and Parameters

PDF Exponential

$X \sim \text{Exp}(\lambda)$

The PDF of an exponentially distributed random variable $X$ is:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$\lambda$ is the mean number of occurrences in one interval (same as Poisson)

Another way of describing the exponential distribution is using $\lambda = \frac{1}{\beta}$

- Continuous random variable
- Models waiting times between events
- Defined only for x>0
- Memoryless property

# Exponential Distribution – CDF

PDF Exponential

$X \sim \text{Exp}(\lambda)$

The PDF of an exponentially distributed random variable $X$ is:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

CDF Exponential

$$F_X(x) = P(X \leq x) = \int_0^x f_X(x)dx = \int_0^x \lambda e^{-\lambda x} dx = - e^{-\lambda x}\Big|_0^x = -e^{-\lambda x} - \left(-e^0\right) = 1 - e^{-\lambda x}$$

# Exponential Distribution – Moments

The mean is given by $E[X] = \frac{1}{\lambda}$, indicating the average time between events.

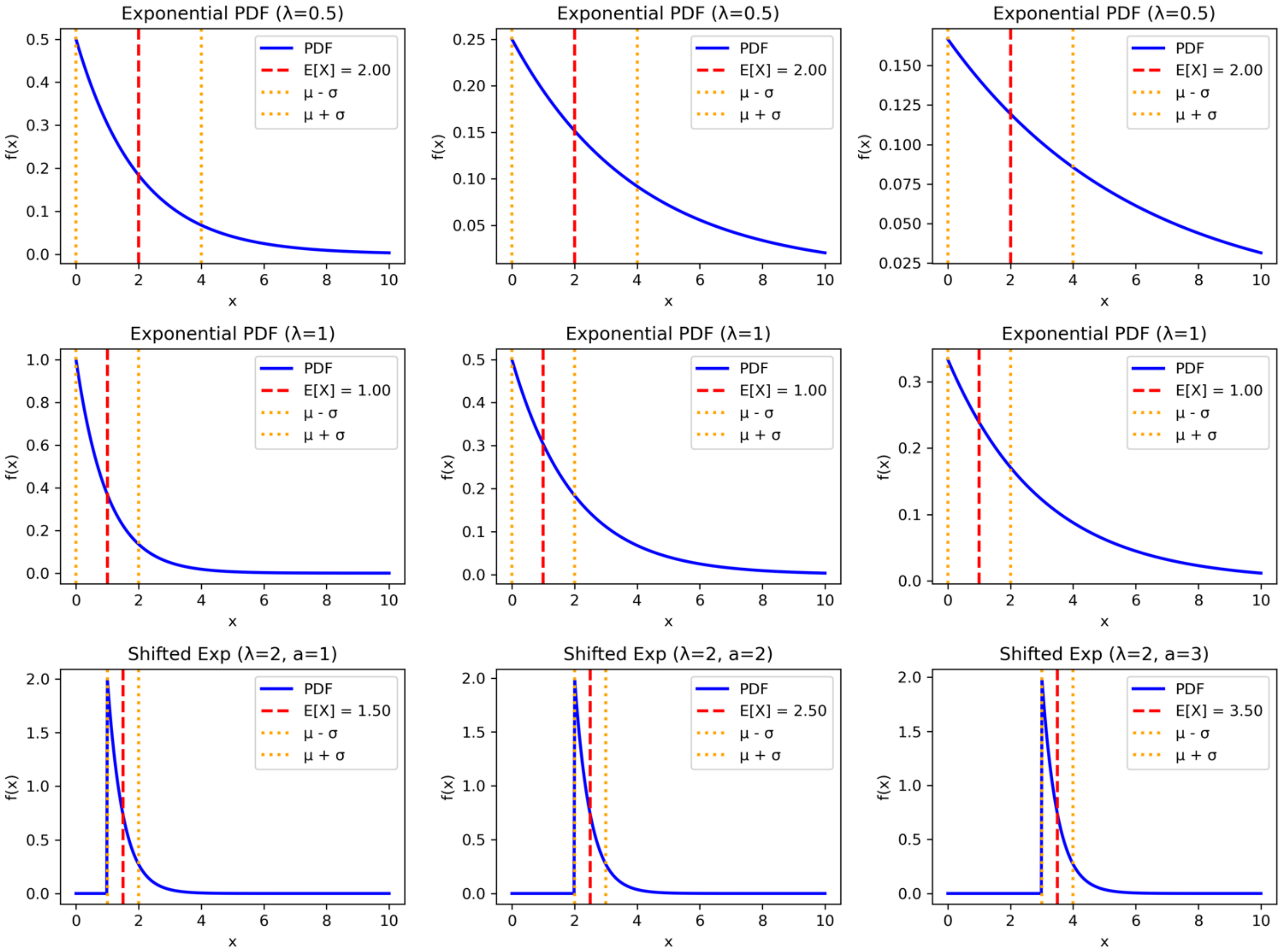
$$E[X] = \frac{1}{\lambda} = \beta$$

The variance is $V[X] = \frac{1}{\lambda^2}$, reflecting the spread of inter-event times.

$$Var[X] = \frac{1}{\lambda^2} = \beta^2$$

# Exponential Distributions

| Parameter | Effect when increased | Effect when decreased |
|---|---|---|
| $\lambda$ (rate) | Steeper decay, shorter waiting times, lower mean and variance | Flatter curve, longer waiting times, higher mean and variance |
| $a$ (shift) | Distribution shifts right (delayed start) | Distribution shifts left (earlier start) |



# Example 1 – Earthquake

There were 16 high intensity earthquakes that occurred in San Francisco between 1836 and 1961. If we assume that the occurrence of such events follows a Poisson process, what is the probability that the next earthquake will occur within 2 years?

Let T be time until the next earthquake

$$T \sim Exp\left(\lambda = \frac{16}{1961-1836} = 0.128\ earthquakes/year\right)$$

$$P(T \leq 2) = 1 - e^{-0.128 \times 2} = 0.226$$

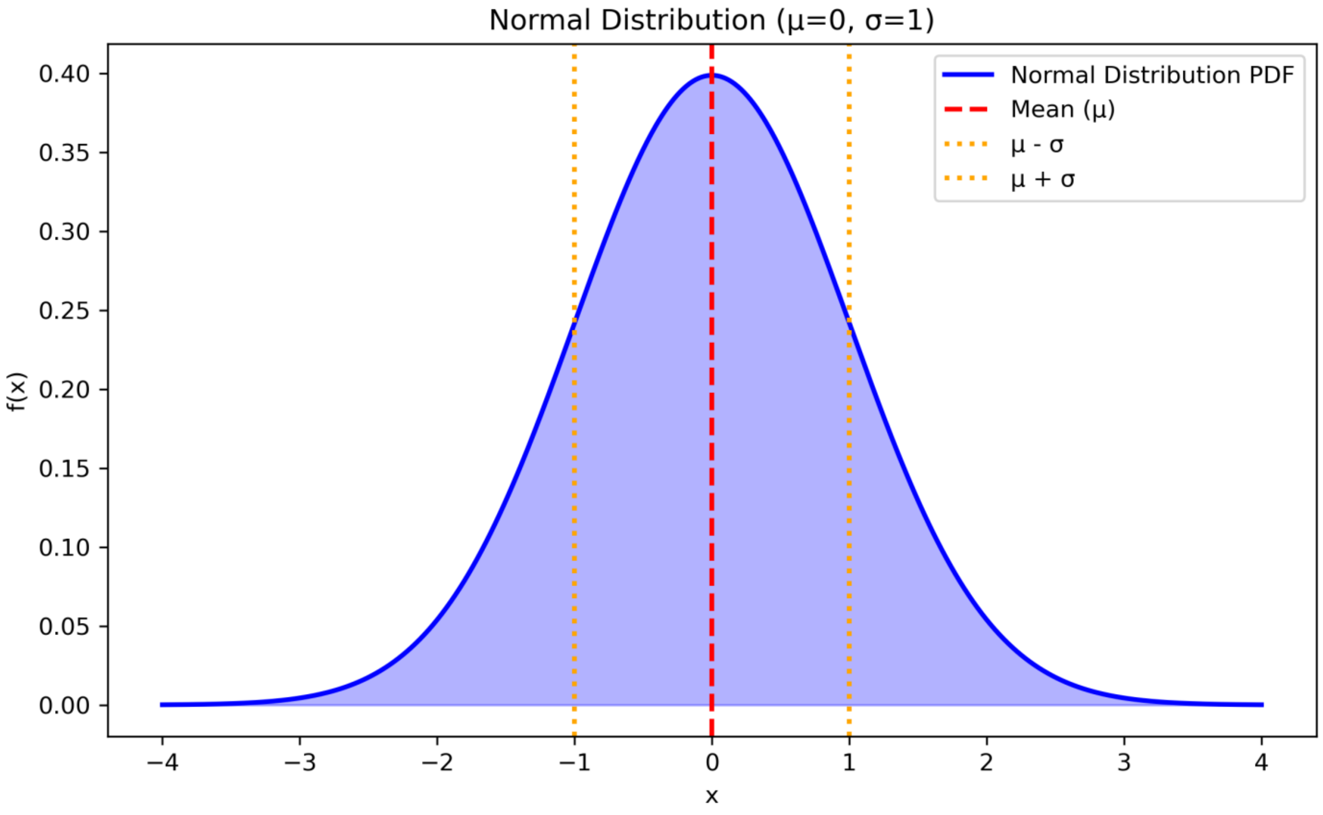
# Differentiate between Poisson and Exponential

| Exponential | Poisson |
|---|---|
| P(it takes 15 minutes for the next call) | P(3 to 5 calls will take place in the next 15 minutes) |
| P(10 years until the next earthquake to occur) | P(1 or more earthquakes in the next 20 years) |

# Normal Distribution – Definition

The best known and most widely used probability distribution.

- Also referred to as **Gaussian distribution**
- Describes a lot of real-world phenomena
- Continuous probability distribution characterized by its **symmetric, bell-shaped curve.**
- It is widely used in statistics, natural sciences, and engineering due to its properties.



# Normal Distribution – PMF and Parameters

The PDF of a normally distributed random variable X is: $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty \leq x \leq \infty$$

- $\mu$ (mean) determines the center of the distribution.
- $\sigma$ (standard deviation) determines the spread of the distribution.
- The normal distribution is bell-shaped and symmetric.
- The total area under the curve equals 1 .

- Continuous random variable
- Symmetric bell curve
- Defined for all real values of x
- Mean = Median = Mode

# Normal Distribution – CDF and Moments

CDF of a normal distribution

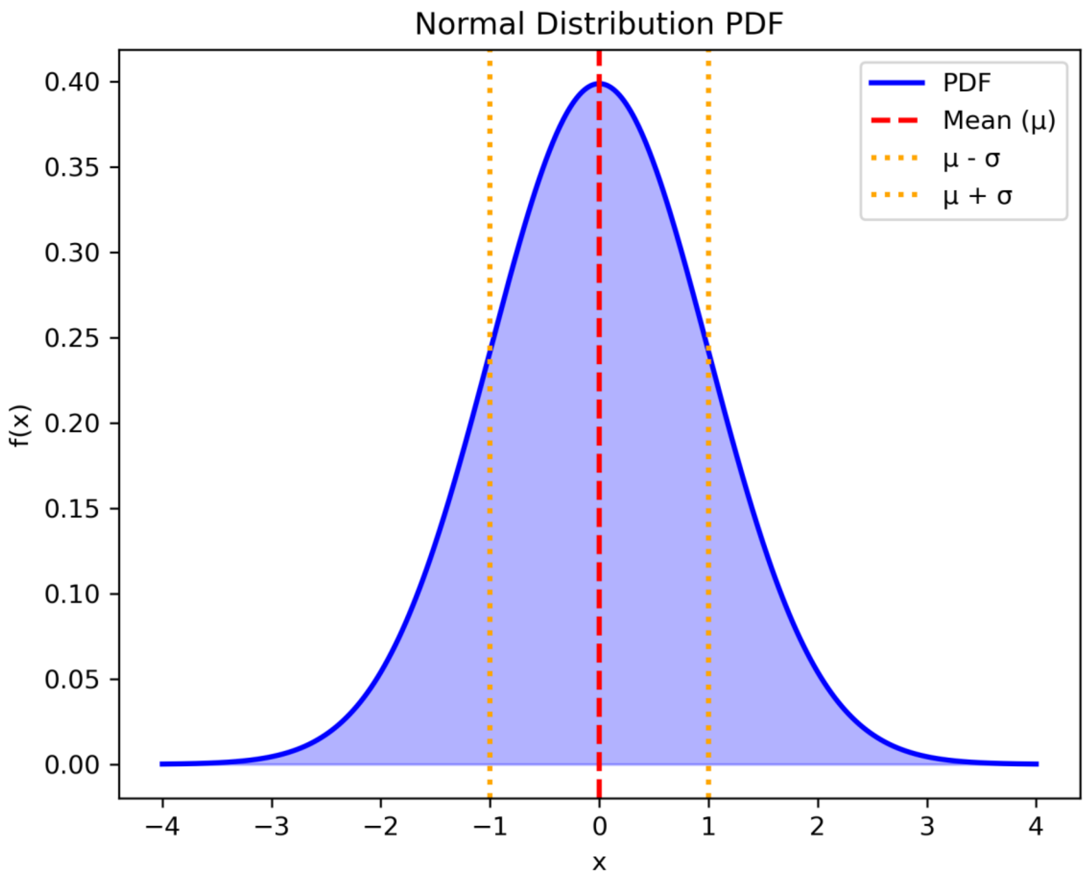
$$F_X(x) = P(a < X \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2} dx$$

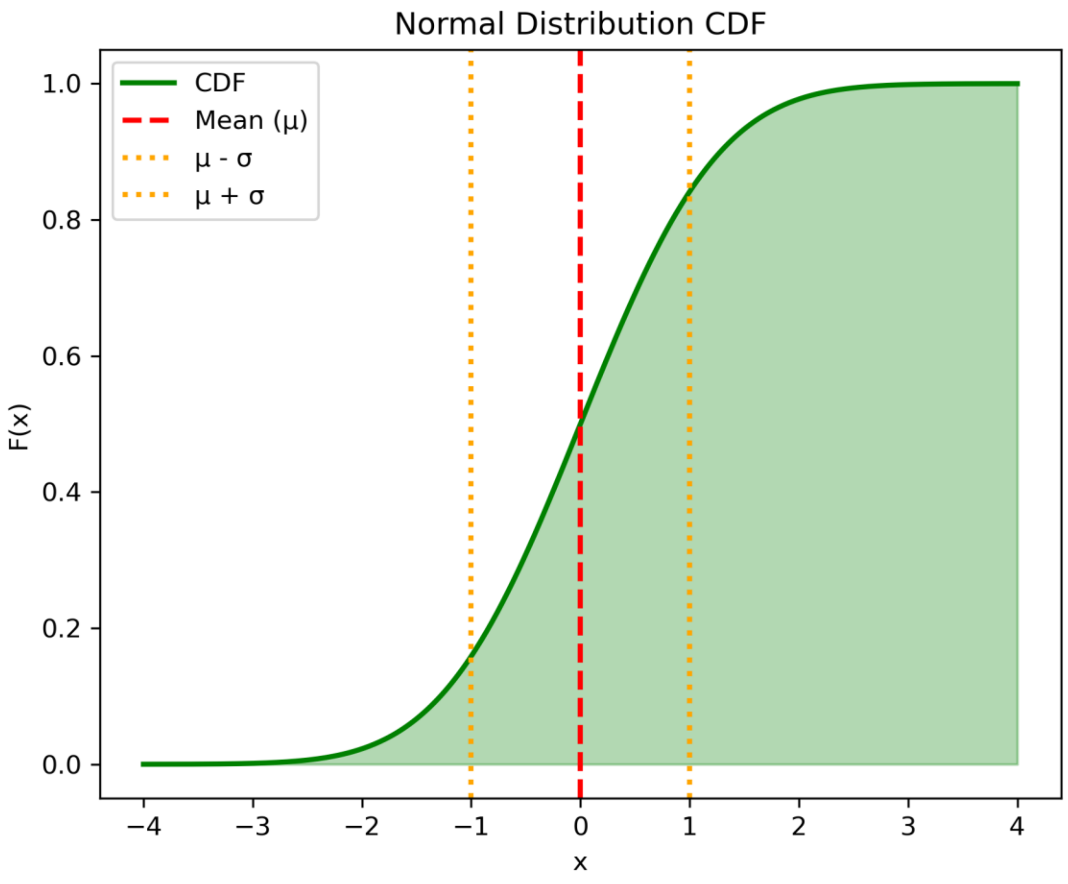
- No closed form
- Use tables to calculate probabilities

The mean and the variance can be calculated to be

$$\mathbf{E}[X] = \mu, \quad \text{var}(X) = \sigma^2$$

# Normal Distribution



# Quiz 2

# Standard Normal Distribution

Let $X$ be a normal random variable with mean $\mu$ and variance $\sigma^2$.

We "standardize" $X$ by defining a new random variable $Y$ given by

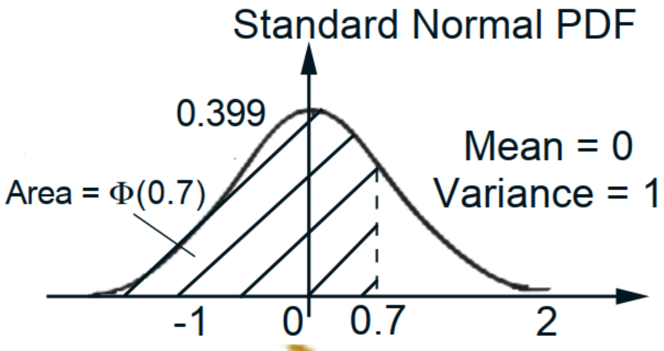
$$Y = \frac{X-\mu}{\sigma} \qquad Y \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$$

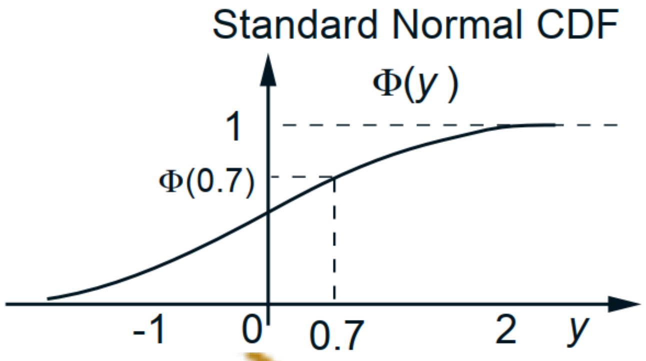
A normal random variable $Y$ with zero mean and unit variance is said to be a standard normal. Its CDF is denoted by $\Phi$,

$$\Phi(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$$

**Complement**

$$\mathbf{P}(Y > y) = 1 - \mathbf{P}(Y \leq y)$$



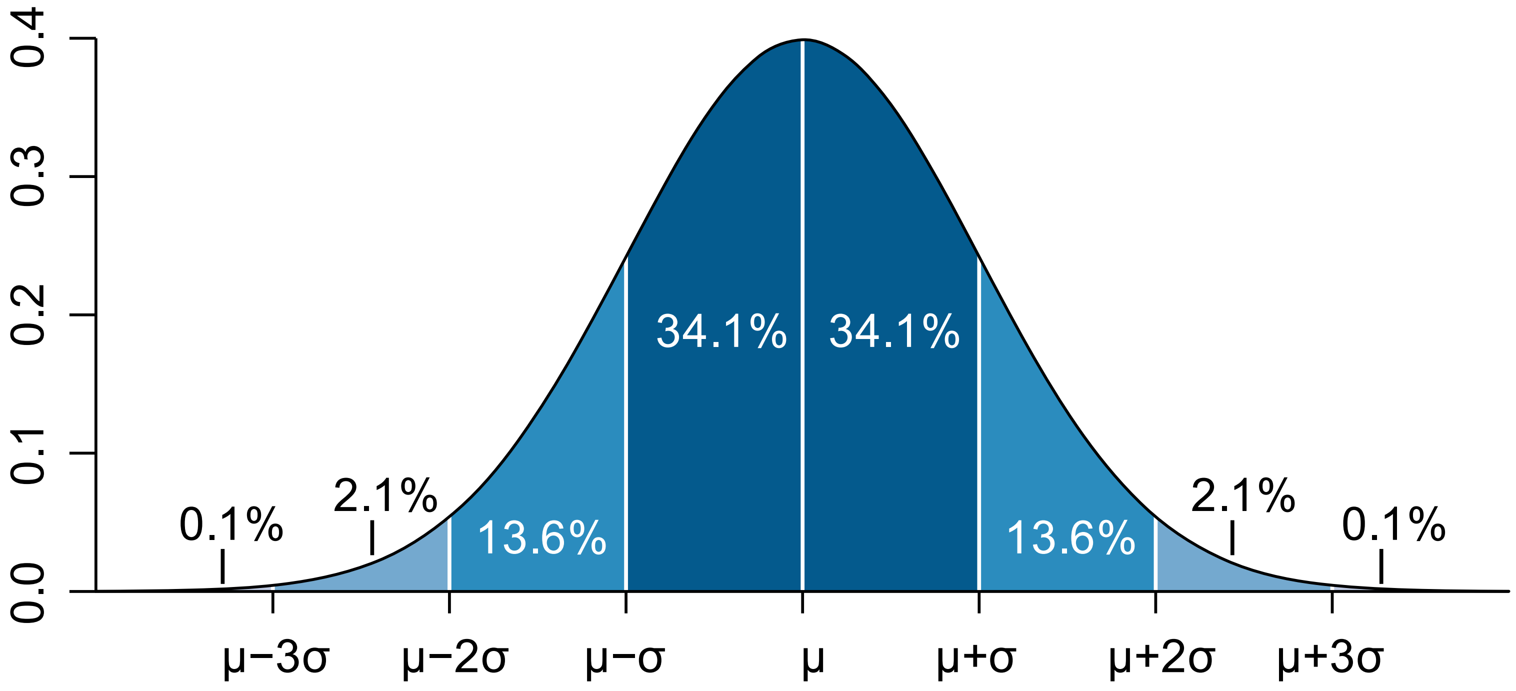

**Negative Values**

$$\Phi(-y) = 1 - \Phi(y)$$

# The 68-95-99.7 Rule

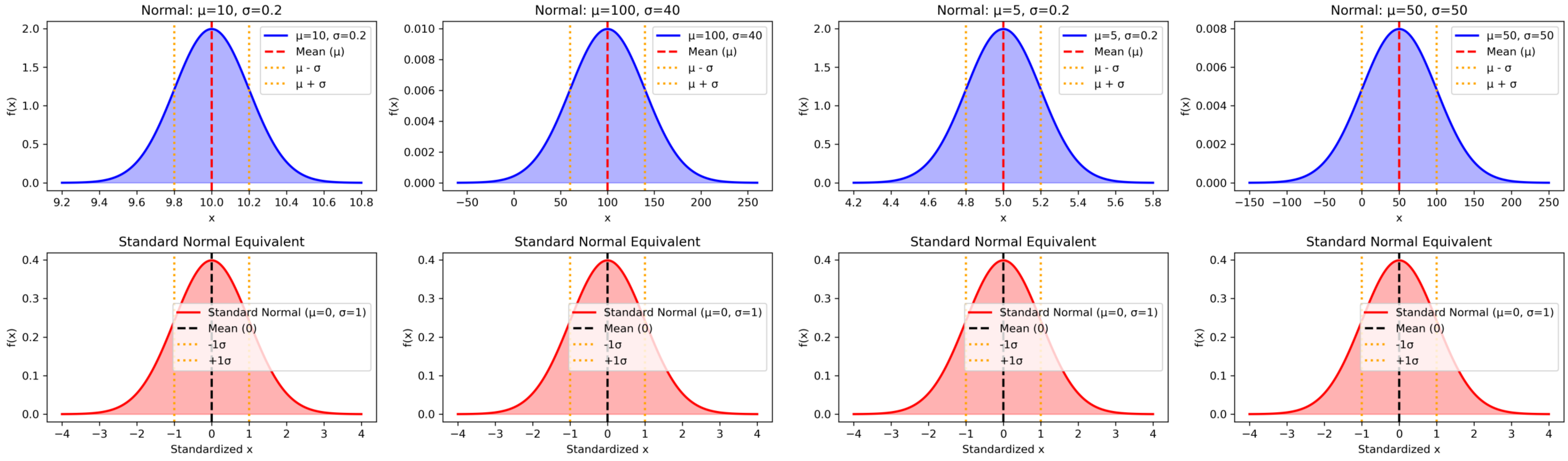
The 68-95-99.7 Rule:

- 68% of data falls within 1 standard deviation ($\mu \pm \sigma$).
- 95% falls within 2 standard deviations ($\mu \pm 2\sigma$).
- 99.7% falls within 3 standard deviations ($\mu \pm 3\sigma$).




$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

# Normal to Standard Normal

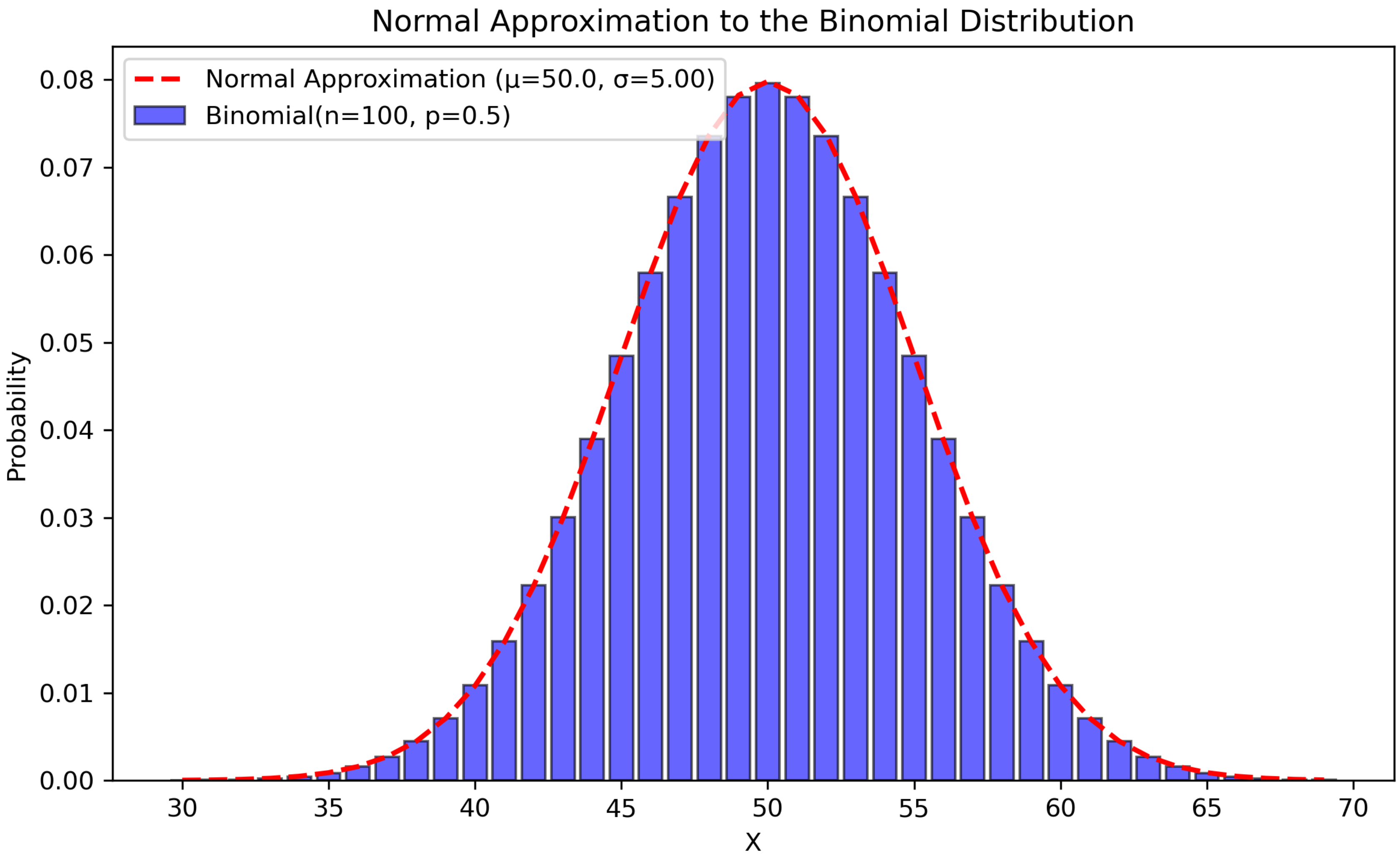


# Recall: Special Cases for Binomial Distribution

$$X \sim \text{binomial}(n, p)$$

$X \sim normal(np, \sqrt{npq})$ if $npq$ not small

$X \sim poisson(\lambda)$ $\lambda = np$ if $np$ small



# Example 2 – Standard Normal

Consider a RV: $Z \sim \text{normal}(0, 1)$
Calculate and draw the following:

$P(Z < 0) = 0.5$

$P(0 < Z < 1.12) = F(1.12) - F(0) = 0.86864 - 0.5 = 0.36864$

**TABLE A.1** Standard Normal Probabilities (pg. 1 of 4) $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$

| $x$ | $\Phi(x)$ | $x$ | $\Phi(x)$ | $x$ | $\Phi(x)$ |
|---|---|---|---|---|---|
| 0.00 | 0.500000000 | 0.40 | 0.655421742 | 0.80 | 0.788144601 |
| 0.01 | 0.503989356 | 0.41 | 0.659097026 | 0.81 | 0.791029912 |
| 0.02 | 0.507978314 | 0.42 | 0.662757273 | 0.82 | 0.793891946 |
| 0.03 | 0.511966473 | 0.43 | 0.666402179 | 0.83 | 0.796730608 |
| 0.04 | 0.515953437 | 0.44 | 0.670031446 | 0.84 | 0.799545807 |
| 0.05 | 0.519938806 | 0.45 | 0.673644780 | 0.85 | 0.802337457 |
| 0.06 | 0.523922183 | 0.46 | 0.677241890 | 0.86 | 0.805105479 |
| 0.07 | 0.527903170 | 0.47 | 0.680822491 | 0.87 | 0.807849798 |
| 0.08 | 0.531881372 | 0.48 | 0.684386303 | 0.88 | 0.810570345 |
| 0.09 | 0.535856393 | 0.49 | 0.687933051 | 0.89 | 0.813267057 |
| 0.10 | 0.539827837 | 0.50 | 0.691462461 | 0.90 | 0.815939875 |
| 0.11 | 0.543795313 | 0.51 | 0.694974269 | 0.91 | 0.818588745 |
| 0.12 | 0.547758426 | 0.52 | 0.698468212 | 0.92 | 0.821213602 |
| 0.13 | 0.551716787 | 0.53 | 0.701944035 | 0.93 | 0.823814458 |
| 0.14 | 0.555670005 | 0.54 | 0.705401484 | 0.94 | 0.826391220 |
| 0.15 | 0.559617692 | 0.55 | 0.708840313 | 0.95 | 0.828943874 |
| 0.16 | 0.563559463 | 0.56 | 0.712260281 | 0.96 | 0.831472393 |
| 0.17 | 0.567494932 | 0.57 | 0.715661151 | 0.97 | 0.833976754 |
| 0.18 | 0.571423716 | 0.58 | 0.719042691 | 0.98 | 0.836456941 |
| 0.19 | 0.575345435 | 0.59 | 0.722404675 | 0.99 | 0.838912940 |
| 0.20 | 0.579259709 | 0.60 | 0.725746882 | 1.00 | 0.841344746 |
| 0.21 | 0.583166163 | 0.61 | 0.729069096 | 1.01 | 0.843752355 |
| 0.22 | 0.587064423 | 0.62 | 0.732371107 | 1.02 | 0.846135770 |
| 0.23 | 0.590954115 | 0.63 | 0.735652708 | 1.03 | 0.848494997 |
| 0.24 | 0.594834872 | 0.64 | 0.738913700 | 1.04 | 0.850830050 |
| 0.25 | 0.598706326 | 0.65 | 0.742153889 | 1.05 | 0.853140944 |
| 0.26 | 0.602568113 | 0.66 | 0.745373085 | 1.06 | 0.855427700 |
| 0.27 | 0.606419873 | 0.67 | 0.748571105 | 1.07 | 0.857690346 |
| 0.28 | 0.610261248 | 0.68 | 0.751747770 | 1.08 | 0.859928910 |
| 0.29 | 0.614091881 | 0.69 | 0.754902906 | 1.09 | 0.862143428 |
| 0.30 | 0.617911422 | 0.70 | 0.758036348 | 1.10 | 0.864333939 |
| 0.31 | 0.621719522 | 0.71 | 0.761147932 | 1.11 | 0.866500487 |
| 0.32 | 0.625515835 | 0.72 | 0.764237502 | 1.12 | 0.868643119 |
| 0.33 | 0.629300019 | 0.73 | 0.767304908 | 1.13 | 0.870761888 |
| 0.34 | 0.633071736 | 0.74 | 0.770350003 | 1.14 | 0.872856849 |
| 0.35 | 0.636830651 | 0.75 | 0.773372648 | 1.15 | 0.874928064 |
| 0.36 | 0.640576433 | 0.76 | 0.776372708 | 1.16 | 0.876975597 |
| 0.37 | 0.644308755 | 0.77 | 0.779350054 | 1.17 | 0.878999516 |
| 0.38 | 0.648027292 | 0.78 | 0.782304562 | 1.18 | 0.880999893 |
| 0.39 | 0.651731727 | 0.79 | 0.785236116 | 1.19 | 0.882976804 |

# Example 2 – Standard Normal

Consider a RV: $Z \sim \text{normal}(0,1)$
Calculate and draw the following:

$$P(-2.17 < Z < 0) = F(0) - F(-2.17) = F(0) - (1 - F(2.17)) = 0.5 - 1 + 0.8577 = 0.48499$$

$$P(Z < -1.17) = 1 - F(1.17) = 1 - 0.87899 = 0.12101$$

$$P(-1.56 < Z < 1.31) = F(1.31) - (1 - F(1.56))$$

# Example 3– Normal Distribution

The heights of male adults in the US are approximately normal, with mean 70 in. and standard deviation 3.0 in.

- What is the percentage of male adults in the US who are at least as tall as
  - Kevin Durant, 6′9″ (i.e., 81 in.)
  - Manute Bol, 7′7″ (i.e., 91 in.)
  - Muggsy Bogues, 5′3″ (i.e., 63 in.)

$$X \sim N(\mu, \sigma) \quad Z \sim N(0,1) \quad x \sim N(\mu = 70, \sigma = 3)$$

Kevin: $P(X > 81) = P\left(Z > \frac{(81-70)}{3}\right) = P(Z > 3.67) = 1 - F(3.67) = 1 - 0.999878 = 0.000122$

Manute: $P(X > 91) = P\left(Z > \frac{(91-70)}{3}\right) = P(Z > 7) = 1-F(7) = 1.28e-12$

Muggsy: $P(X > 63) = P\left(Z > \frac{(63-70)}{3}\right) = P(Z > 2.3) = 1 - F(2.3) = 1 - 0.98927 = 0.01073$

# Example 3 – Continued

The heights of male adults in the US are approximately normal, with mean 70 in. and standard deviation 3.0 in.

- What is the percentage of male adults in the US who are at least as tall as
  - Kevin Durant, 6′9″ (i.e., 81 in.)
  - Manute Bol, 7′7″ (i.e., 91 in.)
  - Muggsy Bogues, 5′3″ (i.e., 63 in.)

If there are approximately 100 million male adults in the US, roughly how many in the population does the above relate to?

Kevin: 12,200 are at least as tall as KD

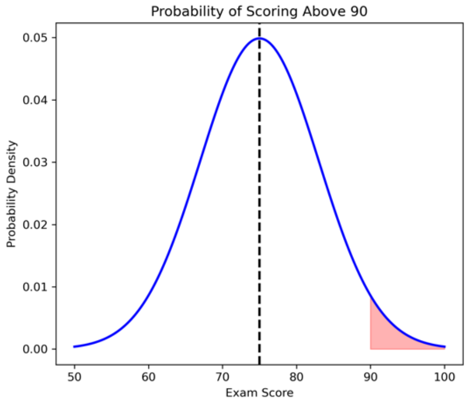
Manute: 0.000128

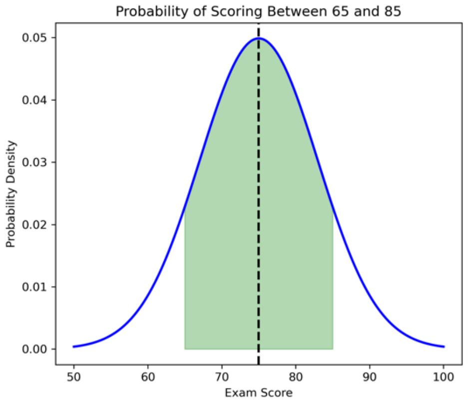
Muggsy: 1,073,000 as tall as Manute

# Example 4 – Exam Scores

A university professor gives an exam where the scores are **normally distributed** with **mean** of 75 and **standard deviation** of 8.

a. What percentage of students score **above 90**?
b. What proportion of students score **between 65 and 85** ?




Let $X \sim \mathcal{N}\left(\mu = 75, \sigma^2 = 64\right)$

Then, $Z \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$

a. $z = \frac{x-\mu}{\sigma} = \frac{90-75}{8} = 1.875 \quad \rightarrow P(Z > 1.875) = 1 - P(Z < 1.875) = 1 - 0.9693 = 0.0307$

b. $P(65 < X < 85) = P(-1.25 < Z < 1.25) = \Phi(1.25) - \Phi(-1.25)$

$$= 0.8944 - (1 - 0.8944) = 0.8944 - 0.1056 = 0.7888$$

# Example 4 – Lifespan of Lightbulbs

A company manufactures lightbulbs, and their lifespan follows a normal distribution with mean of 1000 hours and standard deviation of 200 hours.

a. **What is the probability that a randomly selected lightbulb lasts less than 800 hours?**
b. **What should the warranty period be if the company wants to cover only the bottom 5% of bulbs?**

Let $X \sim \mathcal{N}\left(\mu = 1000, \sigma^2 = 200^2\right)$

Then, $Z \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$

a. $P(X < 800) = P\left(Z < \frac{800-1000}{200}\right) = P(Z < -1)$

$$= (1 - \Phi(1)) = 1 - 0.8413 = 0.1587$$

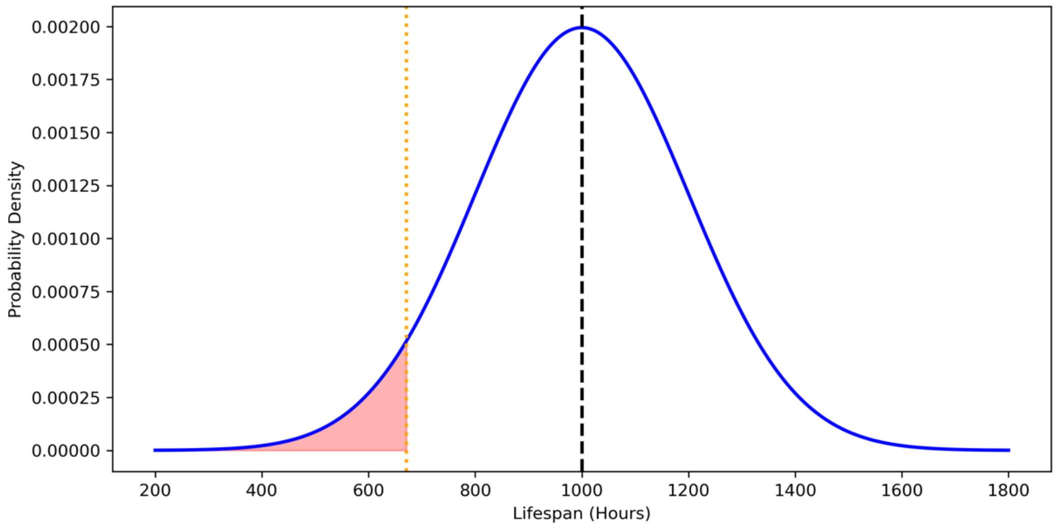
# Example 4 – Lifespan of Lightbulbs

b. **What should the warranty period be if the company wants to cover only the bottom 5% of bulbs?**

We want to find the value of X where 5% of the data is below it.

Let $X \sim \mathcal{N}\left(\mu = 1000, \sigma^2 = 200^2\right)$

Then, $Z \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$

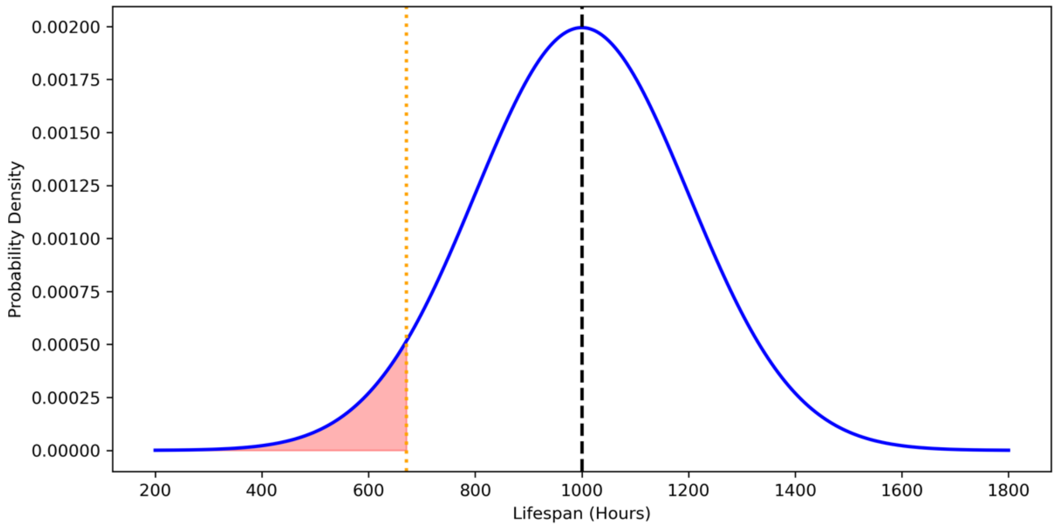



b. We are looking for the value of $P(Z < z) = 0.05$
From the $Z$-table, what is the $Z$-score corresponding to $P(Z) = 0.05$

# Example 4 – Lifespan of Lightbulbs

b. **What should the warranty period be if the company wants to cover only the bottom 5% of bulbs?**

If $P(Z < z)$ is to be 0.05
then, we can say given the symmetry of the normal distribution that $P(Z > z)$ is to be 0.95




Then to find z for the probability of 0.95 we refer to Appendix A.

From Appendix A, for $\Phi(z) = 0.95$, we have $z = 1.645$

we now refer to the relation $\rightarrow \quad \Phi(-y) = 1 - \Phi(y)$

As such, $0.05 = 1 - \Phi(z) = 1 - \Phi(1.645) = \Phi(-1.645)$ $\rightarrow z = -1.645$

Then, $X = \mu + Z\sigma = 1000 + (-1.645)(200) = 670$